# Problems

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**Problem 1** Assuming ideal MHD, prove the *Walen's Equation*.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathbf{B}}{\rho}\right) = \left(\frac{\mathbf{B}}{\rho}\cdot\nabla\right)\mathbf{u}$$

**Solution** WLOG, the quantities $\mathbf{B}$, $\rho$ and $\mathbf{u}$ correspond to magnetic field, density and fluid velocity respectively.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathbf{B}}{\rho}\right) &= \frac{\partial}{\partial t}\left(\frac{\mathbf{B}}{\rho}\right) + \mathbf{u}\cdot\nabla\left(\frac{\mathbf{B}}{\rho}\right)\\ &= \left\{\frac{1}{\rho}\frac{\partial\mathbf{B}}{\partial t} - \frac{\mathbf{B}}{\rho^2}\frac{\partial\rho}{\partial t}\right\} + \mathbf{u}\cdot\nabla\left(\frac{\mathbf{B}}{\rho}\right)\end{aligned}$$

Recall, continuity equation

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0$$

$$\nabla\cdot(\rho\mathbf{u}) = \nabla\rho\cdot\mathbf{u} + \rho(\nabla\cdot\mathbf{u})$$

and induction equation,

$$\frac{\partial\mathbf{B}}{\partial t} = \nabla\times(\mathbf{u}\times\mathbf{B})$$

Substituting these,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathbf{B}}{\rho}\right) = \left\{\frac{1}{\rho}\nabla\times(\mathbf{u}\times\mathbf{B}) + \mathbf{B}\left(\frac{\nabla\rho\cdot\mathbf{u}}{\rho^2} + \frac{(\nabla\cdot\mathbf{u})}{\rho}\right)\right\} + \mathbf{u}\cdot\nabla\left(\frac{\mathbf{B}}{\rho}\right)$$

Using the vector identity

$$\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - \mathbf{B}(\nabla\cdot\mathbf{A}) - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B})$$

Substituting $\mathbf{A} = \mathbf{u}$,

$$\nabla\times(\mathbf{u}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{u} - \mathbf{B}(\nabla\cdot\mathbf{u}) - (\mathbf{u}\cdot\nabla)\mathbf{B} + \cancel{\mathbf{u}(\nabla\cdot\mathbf{B})}$$

Since $\nabla\cdot\mathbf{B} = 0$ , the final term vanishes. Therefore,

$$\frac{1}{\rho}\nabla\times(\mathbf{u}\times\mathbf{B}) = \frac{1}{\rho}(\mathbf{B}\cdot\nabla)\mathbf{u} - \frac{\mathbf{B}}{\rho}(\nabla\cdot\mathbf{u}) - \frac{1}{\rho}(\mathbf{u}\cdot\nabla)\mathbf{B}$$

Now we shall expand the Advection Term $\mathbf{u}\cdot\nabla\left(\frac{\mathbf{B}}{\rho}\right)$.
Using the scalar-vector product rule for the gradient operator,

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$$\begin{aligned}\mathbf{u}\cdot\nabla\left(\frac{\mathbf{B}}{\rho}\right) &= \frac{1}{\rho}(\mathbf{u}\cdot\nabla)\mathbf{B} + \mathbf{B}\left[\mathbf{u}\cdot\nabla\left(\frac{1}{\rho}\right)\right] \\ &= \frac{1}{\rho}(\mathbf{u}\cdot\nabla)\mathbf{B} - \mathbf{B}\left(\frac{\nabla\rho\cdot\mathbf{u}}{\rho^2}\right)\end{aligned}$$

Now, substituting the expansions from previous steps back into the starting equation-

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathbf{B}}{\rho}\right) &= \left\{\frac{1}{\rho}(\mathbf{B}\cdot\nabla)\mathbf{u} - \frac{\mathbf{B}}{\rho}(\nabla\cdot\mathbf{u}) - \frac{1}{\rho}(\mathbf{u}\cdot\nabla)\mathbf{B} + \mathbf{B}\left(\frac{\nabla\rho\cdot\mathbf{u}}{\rho^2} + \frac{\nabla\cdot\mathbf{u}}{\rho}\right)\right\} \\ &+ \frac{1}{\rho}(\mathbf{u}\cdot\nabla)\mathbf{B} - \mathbf{B}\left(\frac{\nabla\rho\cdot\mathbf{u}}{\rho^2}\right) \\ &= \left\{\frac{1}{\rho}(\mathbf{B}\cdot\nabla)\mathbf{u} - \frac{\mathbf{B}}{\rho}(\nabla\cdot\mathbf{u}) - \frac{1}{\rho}(\mathbf{u}\cdot\nabla)\mathbf{B} + \mathbf{B}\left(\frac{\nabla\rho\cdot\mathbf{u}}{\rho^2}\right) + \frac{\mathbf{B}}{\rho}(\nabla\cdot\mathbf{u})\right\} \\ &+ \frac{1}{\rho}(\mathbf{u}\cdot\nabla)\mathbf{B} - \mathbf{B}\left(\frac{\nabla\rho\cdot\mathbf{u}}{\rho^2}\right) \\ &= \frac{1}{\rho}(\mathbf{B}\cdot\nabla)\mathbf{u} - \cancel{\frac{\mathbf{B}}{\rho}(\nabla\cdot\mathbf{u})} + \cancel{\frac{\mathbf{B}}{\rho}(\nabla\cdot\mathbf{u})} \\ &- \cancel{\frac{1}{\rho}(\mathbf{u}\cdot\nabla)\mathbf{B}} + \cancel{\frac{1}{\rho}(\mathbf{u}\cdot\nabla)\mathbf{B}} + \cancel{\mathbf{B}\left(\frac{\nabla\rho\cdot\mathbf{u}}{\rho^2}\right)} - \cancel{\mathbf{B}\left(\frac{\nabla\rho\cdot\mathbf{u}}{\rho^2}\right)}\end{aligned}$$

Every single term on the right-hand side cancels out perfectly except for the first term. Since $\rho$ is a scalar quantity, it groups directly into the dot product operator.

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathbf{B}}{\rho}\right) = \left(\frac{\mathbf{B}}{\rho}\cdot\nabla\right)\mathbf{u}}$$

□

**Problem 2**

a) The Bernoulli parameter along a streamline $\frac{v^2}{2} + \int \frac{\mathrm{d}P}{\rho} + \Phi$ is a conserved quantity in the absence of any dissipation. Show that for an adiabatic flow (EOS $P = K\rho^\gamma$ with K being a constant), one can obtain a constant of motion as

$$\varepsilon = \frac{v^2}{2} + na^2 + \Phi,$$

where, $n = 1/(\gamma - 1)$ and $a$ represents the sound speed.

**Solution** We first show that the Bernoulli Parameter is indeed a conserved quantity. Recall, from Euler's equation (in the absence of any dissipative term)

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = -\frac{1}{\rho}\nabla P + \frac{\mathbf{F}}{m}$$

We seek to find steady state solutions, i.e $\frac{\partial \boldsymbol{v}}{\partial t} = 0$. Use,

$$(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = \nabla\left(\frac{\boldsymbol{v}^2}{2}\right) - \boldsymbol{v}\times(\nabla\times\boldsymbol{v})$$

$$\nabla\left(\frac{\boldsymbol{v}^2}{2}\right) - \boldsymbol{v}\times(\nabla\times\boldsymbol{v}) = -\frac{1}{\rho}\nabla P + \frac{\mathbf{F}}{m}$$

Integrate both sides along a **streamline** ($\boldsymbol{v}\times\mathrm{d}\boldsymbol{\ell} = 0$)

> Note,
>
> $$\boldsymbol{v}\times(\nabla\times\boldsymbol{v})\cdot\mathrm{d}\boldsymbol{\ell} \iff (\nabla\times\boldsymbol{v})\cdot(\boldsymbol{v}\times\mathrm{d}\boldsymbol{\ell}) = 0$$
>
> Thus, the contribution from second term vanishes. Additionally, we define
>
> $$\boldsymbol{f} = \frac{\mathbf{F}}{m}$$
>
> as the force per unit mass.

$$\int_{\boldsymbol{v}\times\mathrm{d}\boldsymbol{\ell}=0} \mathrm{d}\boldsymbol{\ell}\cdot\left[\nabla\left(\frac{\boldsymbol{v}^2}{2}\right) - \cancel{\boldsymbol{v}\times(\nabla\times\boldsymbol{v})} + \frac{1}{\rho}\nabla P - \boldsymbol{f}\right] = C \text{ (constant)}$$

$$\frac{\boldsymbol{v}^2}{2} + \int \frac{1}{\rho}\nabla P\cdot\mathrm{d}\mathbf{l} - \int \boldsymbol{f}\cdot\mathrm{d}\boldsymbol{\ell} = C$$

$$\implies \frac{v^2}{2} + \int \frac{\mathrm{d}P}{\rho} + \Phi = C$$

Hence, the claim in the question is valid (where we have defined $\Phi \equiv -\int \boldsymbol{f}\cdot\mathrm{d}\boldsymbol{\ell}$ as the **potential**). For an adiabatic flow $P = K\rho^\gamma$. So, $\mathrm{d}P = K\gamma\rho^\gamma/\rho\mathrm{d}\rho \implies \mathrm{d}P/\rho = K\gamma\rho^{\gamma-2}\mathrm{d}\rho$.

$$\int \frac{\mathrm{d}P}{\rho} = \int K\gamma\rho^{\gamma-2}\mathrm{d}\rho = \frac{K\gamma\rho^\gamma}{(\gamma-1)\rho} = \frac{1}{(\gamma-1)}\left(\sqrt{\frac{\gamma P}{\rho}}\right)^2 = na^2 + \text{constant}$$

hence,

$$\boxed{\varepsilon \equiv \frac{v^2}{2} + na^2 + \Phi}$$

is indeed a constant of motion.

b) For the adiabatic, radial accretion/wind flow problem onto a Newtonian star ($\Phi = -GM/r$), the mass flux $\dot{M} = \rho v r^2$ is also a conserved quantity. From this, show that one can obtain

$$\dot{\mathcal{M}} = a^{2n} v r^2$$

as another constant of motion for such flow, in addition to $\varepsilon$.

**Solution** From definition of sound speed,

$$a^2 \equiv \frac{\mathrm{d}P}{\mathrm{d}\rho}$$

We have adiabatic equation of state, $P = K\rho^\gamma$. So,

$$a^2 = K\gamma\rho^{\gamma-1} \implies a^2 \propto \rho^{1/n} \implies \rho \propto a^{2n}$$

Substitute this in mass flux .

$$\rho v r^2 \propto a^{2n} v r^2 = \dot{\mathcal{M}}$$

which is a conserved quantity, as required.

□

c) For the transonic accretion/wind flow, at the critical point, $v_c$, $r_c$ and $a_c$ are related as follows:

$$v_c = a_c, \qquad r_c = \frac{GM}{2a_c^2}.$$

Argue that if such transonic condition is imposed, such a flow cannot have two independent parameters $\varepsilon$ and $\dot{\mathcal{M}}$ determining the unique transonic solution, but only one of these can be the independent parameter.

**Solution** At the critical point (denoted by the subscript $c$), the flow must transition smoothly from subsonic to supersonic (or vice-versa). This transition imposes specific conditions on the flow velocity ($v_c$) and the gravitational potential. The given conditions are:

- $v_c = a_c$
- $r_c = \frac{GM}{2a_c^2}$

Let us apply these conditions to the two conserved quantities $\varepsilon$ and $\dot{\mathcal{M}}$.

$$\begin{aligned}\varepsilon &= \frac{v_c^2}{2} + na_c^2 + \Phi_c = \frac{a_c^2}{2} + na_c^2 - \frac{GM}{r_c} \\ &= \frac{a_c^2}{2} + na_c^2 - 2a_c^2 = \left(n - \frac{3}{2}\right)a_c^2\end{aligned}$$

$$\begin{aligned}\dot{\mathcal{M}} &\propto a_c^{2n} v_c r_c^2 = a_c^{2n+1}\left(\frac{GM}{2a_c^2}\right)^2 \\ &\propto a_c^{2n-3}\end{aligned}$$

This shows that both $\varepsilon$ and $\dot{M}$ are solely dependent on the same variable $a_c$! Thus if a transonic condition is imposed, such a flow cannot have two independent parameters $\varepsilon$ and $\dot{\mathcal{M}}$ determining the unique transonic solution, but only one of these can be the independent parameter.

□

## Numerical Problem

**Problem 3** Find out the solution of spherical accretion/wind using adiabatic equation of state. These are famous Bondi accretion and Parker wind solutions. Note that for this case, the sound speed is not a constant but varies with radius (unlike isothermal case). First perform the sonic point analysis and then proceed to the solution (use numerical method if required!). Draw a plot showing the radial profile of the Mach number, and indicate the accretion and the wind solutions.
Discuss, at least qualitatively, any similarity/dissimilarity with the solution for isothermal case.

**Solution** We assume, the plasma expands adiabatically and is in a steady state. Let the speed of sound be $a(r)$. The equation of state is $P = K\rho^\gamma$. So, we can write,

$$a^2 = \frac{\mathrm{d}P}{\mathrm{d}\rho} = \frac{\gamma P(r)}{\rho(r)} \tag{1}$$

Define Mach Number as

$$\mathcal{M} \equiv \frac{v(r)}{a(r)}$$

We start with 3D steady state hydrodynamic equations with spherical symmetry. We restrict our attention to *spherically symmetric solution*. The velocity is taken as purely radial $\boldsymbol{v} = v\hat{r}$, and the gravitational acceleration $\mathbf{g} = g\hat{r}$, obeys *inverse square law* $\mathbf{g} = -\frac{GM}{r^2}\hat{r}$.

$$\cancel{\frac{\partial \rho}{\partial t}} + \nabla\cdot(\rho\boldsymbol{v}) = 0 \tag{2}$$

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(\rho v r^2\right) = 0 \implies \rho v r^2 = \text{ constant (Mass Flux!)} \tag{3}$$

$$\frac{1}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}r} + \frac{2}{r} + \frac{1}{v}\frac{\mathrm{d}v}{\mathrm{d}r} = 0 \tag{4}$$

and,

$$\rho\cancel{\frac{\partial \boldsymbol{v}}{\partial t}} + \rho(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = -\nabla P + \rho\mathbf{g} \tag{5}$$

$$v\frac{\mathrm{d}v}{\mathrm{d}r} = -\frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r} - \frac{GM}{r^2} \tag{6}$$

Using Eq. (1),

$$\frac{\mathrm{d}P}{\mathrm{d}r} = a^2\frac{\mathrm{d}\rho}{\mathrm{d}r}$$

Substitute this back in Eq. (6) and use Eq. (4),

$$v\frac{\mathrm{d}v}{\mathrm{d}r} = -\frac{a^2}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}r} - \frac{GM}{r^2} \tag{7}$$

$$= a^2\left[\frac{2}{r} + \frac{1}{v}\frac{\mathrm{d}v}{\mathrm{d}r}\right] - \frac{GM}{r^2} \tag{8}$$

Rearranging,

$$\boxed{\frac{1}{v}\frac{\mathrm{d}v}{\mathrm{d}r}\left[\frac{v^2}{a^2} - 1\right] = -\frac{GM}{a^2r^2}\left[1 - \frac{2a^2r}{GM}\right]} \tag{9}$$

This is the famous **Bondi Equation** . It represents a spherically symmetric, steady state adiabatic accretion of a non-self-gravitating gas [2].

[2] Bondi, 1952, MNRAS, 112, 195

Eq. (9) can be also rewritten as,

$$\frac{\mathrm{d}v^2}{\mathrm{d}r} = \frac{\frac{-2GM}{r^2}\left(1-\frac{2ra^2}{GM}\right)}{\left(1-\frac{a^2}{v^2}\right)} \iff \frac{\mathrm{d}v}{\mathrm{d}r} = \frac{\frac{-GM}{vr^2}\left(1-\frac{2ra^2}{GM}\right)}{\left(1-\frac{a^2}{v^2}\right)} \tag{10}$$

**Sonic Point Analysis** Notice, that when the flow velocity reaches the sound speed (at a radius $r_c$),

$$\boxed{v(r_c) = a(r_c) = a_c,}$$

The denominator in Eq. (10), becomes 0. Thus, in order to have a finite and differentiable velocity profile (physically demanded) at sonic point, we require the numerator to also vanish!

$$\boxed{r_c = \frac{GM}{2a_c^2}}$$

This set of points ($r = r_c, a = a_c$) are called **critical points**. Let us evaluate the derivative at critical point, using L'Hôpital's Rule.
Let $N(r)$ be the numerator and $D(r)$ be the denominator. According to this rule, the limit at the critical point can be evaluated by differentiating both the numerator and the denominator with respect to $r$,

$$\left(\frac{\mathrm{d}v^2}{\mathrm{d}r}\right)_{r=r_c} = \lim_{r\to r_c} \frac{\frac{\mathrm{d}N}{\mathrm{d}r}}{\frac{\mathrm{d}D}{\mathrm{d}r}}$$

First, have a look at the numerator $N(r)$

$$\begin{aligned}
N(r) &= -\frac{2GM}{r^2} + \frac{4a^2}{r} \\
\frac{\mathrm{d}N}{\mathrm{d}r} &= \frac{4GM}{r^3} - \frac{4a^2}{r^2} \\
\therefore \left(\frac{\mathrm{d}N}{\mathrm{d}r}\right)_{r=r_c} &= \frac{4GM}{\left(\frac{GM}{2a_c^2}\right)^3} - \frac{4a_c^2}{\left(\frac{GM}{2a_c^2}\right)^2} \\
&= \frac{4GM\cdot 8a_c^6}{(GM)^3} - \frac{4a_c^2\cdot 4a^4}{(GM)^2} \\
&= \frac{32a_c^6}{(GM)^2} - \frac{16a_c^6}{(GM)^2} = \frac{16a_c^6}{(GM)^2} \ \checkmark
\end{aligned}$$

Now, let's attack the denominator $D(r)$. The denominator is $D = 1 - \frac{a^2}{v^2}$. Since $v^2$ is an implicit function of $r$, we apply the chain rule:

$$\frac{\mathrm{d}D}{\mathrm{d}r} = \frac{\mathrm{d}D}{\mathrm{d}(v^2)} \cdot \frac{\mathrm{d}v^2}{\mathrm{d}r} = \frac{a^2}{(v^2)^2} \cdot \frac{\mathrm{d}v^2}{\mathrm{d}r}$$

Evaluating this at the critical point where $v_c^2 = a_c^2$:

$$\left(\frac{\mathrm{d}D}{\mathrm{d}r}\right)_{r=r_c} = \frac{a^2}{(a^2)^2} \cdot \left(\frac{\mathrm{d}v^2}{\mathrm{d}r}\right)_{r=r_c} = \frac{1}{a_c^2}\left(\frac{\mathrm{d}v^2}{\mathrm{d}r}\right)_{r=r_c} \ \checkmark$$

Let $\lambda = \left(\frac{\mathrm{d}v^2}{\mathrm{d}r}\right)_{r=r_c}$ represent the value of the derivative at the critical point. Substituting the derivatives of $N(r)$ , $D(r)$ back into the original expression gives,

$$\lambda = \frac{\frac{16a_c^6}{(GM)^2}}{\frac{1}{a_c^2}\lambda}$$

$$\lambda^2 = \frac{16a_c^8}{(GM)^2}$$

$$\implies \lambda = \pm\frac{4a_c^4}{GM}$$

$$\boxed{\left(\frac{\mathrm{d}v^2}{\mathrm{d}r}\right)_{r=r_c} = \pm\frac{4a_c^4}{GM} \iff \left(\frac{\mathrm{d}v}{\mathrm{d}r}\right)_{r=r_c} = \pm\frac{2a_c^3}{GM}}$$

**This implies that for a curve crossing the sonic point, there are maximum two solutions possible with two different slopes**

Recall, for an adiabatic flow the following expression is a constant of motion (in the presence of a newtonian gravity field)

$$\varepsilon \equiv \frac{v(r)^2}{2} + \frac{a(r)^2}{\gamma - 1} - \frac{GM}{r^2}$$

At sonic point,

$$\begin{aligned}\varepsilon &= \frac{a_c^2}{2} + \frac{a_c^2}{\gamma - 1} - \frac{GM}{r_c} \\ &= \frac{a_c^2}{2} + \frac{a_c^2}{\gamma - 1} - 2a_c^2 \\ &= a_c^2\left[\frac{5 - 3\gamma}{2(\gamma - 1)}\right]\end{aligned}$$

Hence, the family of **transonic solutions** (crossing the sonic point) are given by

$$\boxed{\frac{v(r)^2}{2} + \frac{a(r)^2}{\gamma - 1} - \frac{GM}{r^2} = a_c^2\left[\frac{5 - 3\gamma}{2(\gamma - 1)}\right]} \tag{11}$$

Define dimensionless variables,

$$u(r) = v(r)/a_c \ ; \ x = r/r_c$$

From previous question, recall we have showed $a^{2n}vr^2$ is also a conserved quantity. Therefore,

$$a^{2/(\gamma-1)}vr^2 = a_c^{2/(\gamma-1)}a_c r_c^2$$

We get the following relation,

$$a^2(r) = a_c^2 u^{1-\gamma} x^{-2(\gamma-1)} \tag{12}$$

Using the above expression in Eq (11),

$$\frac{(v/a_c)^2}{2} + \frac{(a/a_c)^2}{\gamma - 1} - \frac{GM}{a_c^2 r} = \left[\frac{5 - 3\gamma}{2(\gamma - 1)}\right] \tag{13}$$

$$\frac{u^2}{2} + \frac{u^{1-\gamma}x^{-2(\gamma-1)}}{\gamma - 1} = \left[\frac{5 - 3\gamma}{2(\gamma - 1)}\right] + \frac{2}{x} \tag{14}$$

The final form gives us the family of all $u(x)$, with parameter $\gamma$.

**Constraint on $\gamma$**

Now let us investigate the conditions on $\gamma$ for which we must have a *real* solution profiles. The lower bound is clearly given by positive polytropic index, i.e $n > 0 \implies \boxed{\gamma > 1}$. $\gamma = 1$ will yield back the **isothermal solution**. Further, if $\gamma < 1$, it implies a negative polytropic index. Physically, compressing this gas would decrease it's pressure and temperature, rather than increase. Such a fluid is unstable to hydrodynamic perturbations and cannot sustain a *steady-state* transonic profile.

Let us define the function

$$F(x,u|\gamma) \equiv \frac{u^2}{2} + \frac{u^{1-\gamma}x^{-2(\gamma-1)}}{\gamma-1} - \left[\frac{5-3\gamma}{2(\gamma-1)}\right] - \frac{2}{x}$$

Along a physical solution curve, the total energy is conserved, meaning $F(x,u) = 0$ is constant everywhere. So, our target is to find a constraint on the parameter $\lambda$, such that $F(x,u) = 0$ gives $u(x) \in \mathbb{R}\ \forall x > 0$. Since sonic/critical point $(x_c, u_c) = (1,1)$ is common point in all the curves,

$$F(x_c, u_c) = 0$$

Furthermore, by definition, a critical point in these hydrodynamic equations is a location where the first derivatives vanish simultaneously.

$$F_x|_{(x_c,u_c)} = F_u|_{(x_c,u_c)} = 0$$

Now calculate the second-order partial derivatives evaluated at critical point $(1,1)$,

$$F_{uu} = \frac{\partial^2 F}{\partial u^2} = 1 + \gamma u^{-\gamma-1}x^{-2(\gamma-1)} \implies F_{uu}|_{(1,1)} = 1+\gamma$$

$$F_{ux} = \frac{\partial^2 F}{\partial u \partial x} = 2(\gamma-1)u^{-\gamma}x^{-2\gamma+1} \implies F_{ux}|_{(1,1)} = 2(\gamma-1)$$

$$F_{xx} = \frac{\partial^2 F}{\partial x^2} = -2(-2\gamma+1)u^{1-\gamma}x^{-2\gamma} - \frac{4}{x^3} \implies F_{xx}|_{(1,1)} = 4\gamma - 6$$

Let $\Delta x = x - x_c$ and $\Delta u = u - u_c$ represent small displacements away from the critical point. If we assume $F(x,u)$ is a smooth function, then we can expand it about the critical point

$$F(x,u) = F(x_c,u_c) + (F_x\Delta x + F_u\Delta u) + \frac{1}{2!}\left(F_{xx}(\Delta x)^2 + 2F_{xu}\Delta x\Delta u + F_{uu}(\Delta u)^2\right) + \mathcal{O}(\Delta x^3, \Delta u^3) = 0$$

Neglecting higher order terms

$$0 + F_{xx}(\Delta x)^2 + 2F_{xu}\Delta x\Delta u + F_{uu}(\Delta u)^2 = 0$$

Next, assuming we are looking at a path where $\Delta x \neq 0$, we divide the entire equation by $(\Delta x)^2$

$$F_{xx} + 2F_{xu}\left(\frac{\Delta u}{\Delta x}\right) + F_{uu}\left(\frac{\Delta u}{\Delta x}\right)^2 = 0$$

take, $\lim_{\Delta x\to 0} \Delta u/\Delta x = \mathrm{d}u/\mathrm{d}x = \zeta$. This leads to a quadratic equation in $\zeta$

$$F_{uu}\zeta^2 + 2F_{xu}\zeta + F_{xx} = 0$$

Putting the values of the second derivative,

$$(1+\gamma)\zeta^2 + 4(\gamma-1)\zeta + (4\gamma-6) = 0$$

For real physical curves to cross through this point, the roots of this quadratic equation (the slopes $\zeta$) must be *real and distinct*. This requires the discriminant of the quadratic equation to be strictly positive.

$$b^2 - 4ac = 16(\gamma-1)^2 - 4(1+\gamma)(4\gamma-6) > 0$$
$$\implies \gamma < 5/3$$

Thus, the physical range for steady transonic solutions is strictly restricted to,

$$\boxed{1 < \gamma < 5/3}$$

### Plotting the curves

Let us take $\gamma = 1.4$ as an example. Observe the behaviour of the function $F(x, u)$ for a wide range of fluid velocities (variable $u$), at a fixed position ($x_0 = 0.5$, say) from the star. We know that the possible values of $u$ for the inflow/outflow cases will occur when $F(x_0, u) = 0$.

$$F(x_0, u|\gamma = 1.4) = \frac{u^2}{2} + \frac{u^{-0.4} x_0^{-0.8}}{0.4} - 1 - \frac{2}{x_0}$$

In other words, we seek to find the $u$-values at which the $F(x_0, u)$ curve crosses the X-axis. This is shown in the graph below Fig. (1)

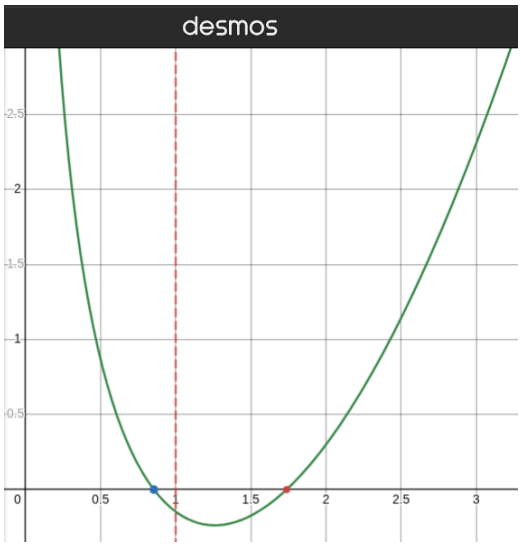



**Figure 1:** A plot of $F(x_0, u|\gamma = 1.4)$ at an arbitrary $x_0 < 1$ (i.e sonic point). One can find two real solutions for the curve, which correspond to the inflow (accretion) or outflow (wind) solutions. As $x_0 \to 1$, we will notice the two roots move closer to each other and converge at $u = 1$ (red dashed vertical line). This is exactly the transonic point that we have reached. Moving past $x_0 > 1$, again two roots separate out (on either side of 1) and diverge.

We determine this *minima* of $F$ as $u_{\text{cross-over}}$. Due to the possibility of only two slopes at critical points, we conclude, that these two curves will intersect at $(1, 1)$ and cross over each other, thereafter. Hence we formulate the following algorithm for plotting the profiles. Let $u_{<1} \leq u_{\text{cross-over}} \leq u_{>1}$ denote the two velocity roots of $F(x_0, u)$, then

$$\boxed{\frac{\partial F}{\partial u} = u - u^{-\gamma} x^{-2(\gamma-1)} = 0 \implies u_{\text{cross-over}} = x^{-\frac{2(\gamma-1)}{\gamma+1}}}$$

(i) If $x_0 < 1 \implies u_{<1} \to u_{\text{wind}}$ and $u_{>1} \to u_{\text{accretion}}$

(ii) If $x_0 \in N_\varepsilon(1)$; $\varepsilon \ll 1$ then $u_{\text{wind}} \approx u_{\text{accretion}} = 1$

(iii) If $x_0 > 1 \implies u_{>1} \to u_{\text{wind}}$ and $u_{<1} \to u_{\text{accretion}}$

### SCRIPT

Here we provide some of the code snippets to do our root finding analysis.

```
import numpy as np
# Suppress warnings
import warnings
warnings.filterwarnings('ignore')

from scipy.optimize import brentq
'''
    Uses the classic Brent's method to find a root of the function `f` on
    the sign changing interval [a , b]. Generally considered the best of the
    rootfinding routines here. It is a safe version of the secant method that
    uses inverse quadratic extrapolation. Brent's method combines root
    bracketing, interval bisection, and inverse quadratic interpolation. It is
    sometimes known as the van Wijngaarden-Dekker-Brent method. Brent (1973)
    claims convergence is guaranteed for functions computable within [a,b].
'''

def F(u, x, gamma):
    # Dimensionless Bondi/Parker equation.
    return (
        0.5*u**2
        + u**(1-gamma)*x**(-2*(gamma-1))/(gamma-1)
        - (5 - 3*gamma)/(2*(gamma-1))
        - 2/x
    )
```

The following function iterates through the radial distances $x \in$ `x_vals` and finds the two valid velocity roots $u$ for each point to construct the Wind and Accretion branches.

```
def solve_transonic_flow(x_vals, gamma):
    # lists to store the values
    wind_u = []
    acc_u  = []

    for x in x_vals:
        # At the sonic point (x=1), the roots converge to exactly u=1.
        if np.isclose(x, 1.0):
            wind_u.append(1.0)
            acc_u.append(1.0)
            continue

        # The equation F(u) = 0 has a single turning point (minimum) with respect to
     u.
        # Setting dF/du = 0 yields this u_cross-over (u_co)

        u_co = x**(-2 * (gamma - 1) / (gamma + 1))

        # Root 1: Subsonic side  (u < 1) (u is between very close to 0 and u_co)
        try:
            u_low = brentq(F, 1e-8, u_co, args=(x, gamma))
        except ValueError:
        # If no roots are found, append nan
            u_low = np.nan

        # Root 2: Supersonic side (u > 1) (u is between u_co and a large number)
        try:
            u_high = brentq(F, u_co, 10e4, args=(x, gamma))
        except ValueError:
            u_high = np.nan

        # Assign roots to physical branches based on the flow topology:
        # 1. Parker Wind: Subsonic (low u) near source x<1, Supersonic (high u) far
    away x>1
        # 2. Bondi Accretion: Supersonic (high u) near source x<1, Subsonic (low u)
    far away x>1
        if x < 1.0:
```

```
            wind_u.append(u_low)
            acc_u.append(u_high)
        else:
            wind_u.append(u_high)
            acc_u.append(u_low)

    return np.array(wind_u), np.array(acc_u)
```

Computing for $\boxed{\gamma = 1.40}$,

```
gamma_val = 1.4

if not (1 < gamma_val < 5/3):
    raise ValueError("gamma must lie b/w  1 and 5/3")

# Generate an array of x = r/r_c values
# using logspace for equal distribution of points across all orders
x_array = np.logspace(-2, 2, 100)

#==========#
# Solve
#==========#
wind_solution, accretion_solution = solve_transonic_flow(x_array, gamma_val)

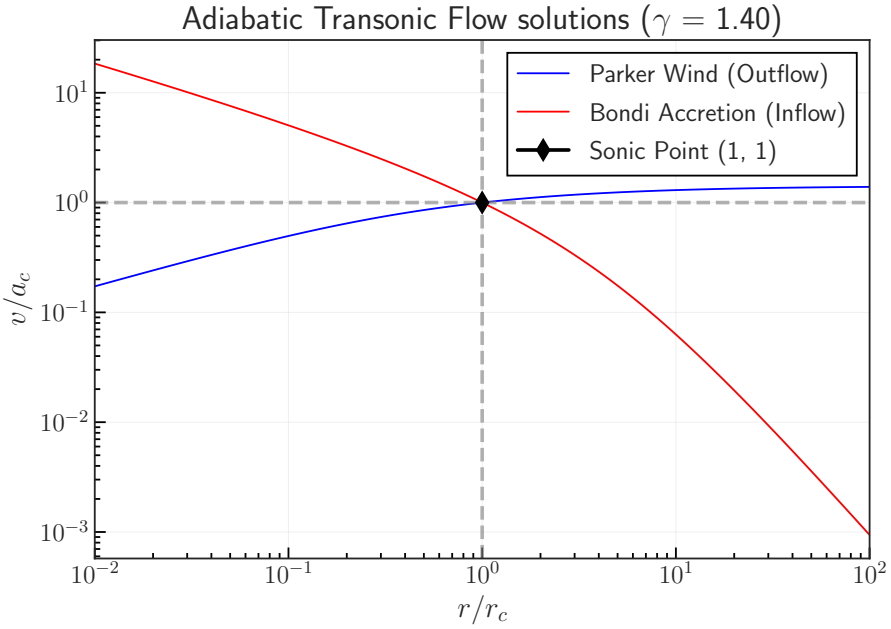
#==========#
# Print Table
#==========#
print(f"--- Transonic Flow Solutions (gamma = {gamma_val}) ---")

table_data = {
    "x (r/r_c)": x_array,
    "Wind u(x)": wind_solution,
    "Accretion u(x)": accretion_solution
}
df = pd.DataFrame(table_data)
# Setting pandas to display the full table
pd.set_option('display.max_rows', None)
pd.set_option('display.float_format', '|{:.4f}|'.format)
print(df.to_string(index=False))
```

The following table is obtained for $x = r/r_c$ and $u(x) = v(r)/a_c$.

| $x$ | $u_{\text{wind}}$ | $u_{\text{accretion}}$ |
|---|---|---|
| 0.0100 | 0.1726 | 18.4380 |
| 0.0110 | 0.1806 | 17.5418 |
| 0.0120 | 0.1889 | 16.6868 |
| ... | ... | ... |
| 83.0218 | 1.3857 | 0.0014 |
| 91.1163 | 1.3875 | 0.0011 |
| 100.0000 | 1.3892 | 0.0009 |

Plotting



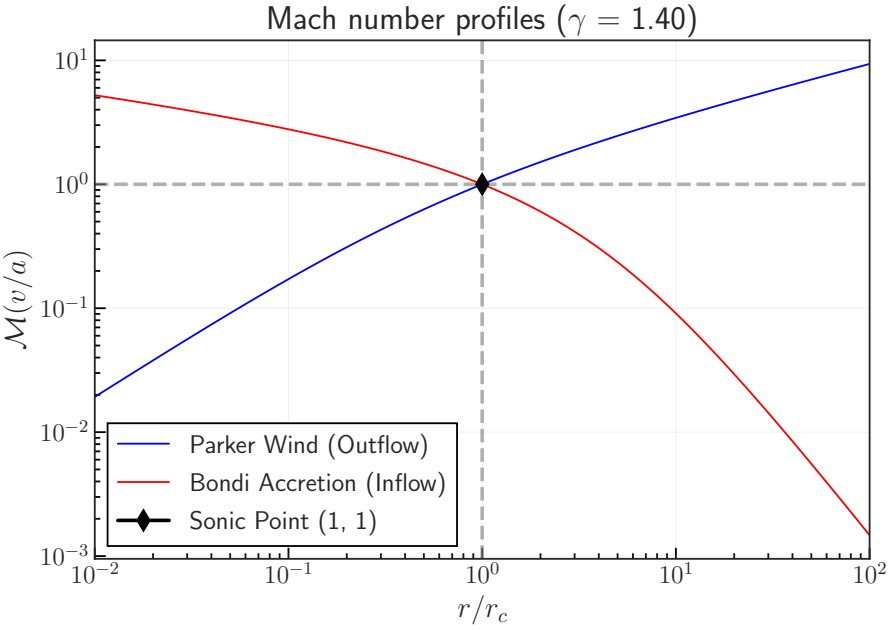

Our final step is to plot the **Mach number profile**, $\mathcal{M} = v(r)/a(r)$ vs $x = r/r_c$. We utilise Eq. (12), to extract $\alpha = a(r)/a_c$. Thus $\mathcal{M} = u/\alpha$

$$a^2(r) = a_c^2 u^{1-\gamma} x^{-2(\gamma-1)}$$

```python
def sound_speed(u, x, gamma):

    # returns a/a_c
    # a_c = sound speed at sonic point
    return np.sqrt( (u ** (1 - gamma)) * x**(-2 * (gamma - 1)))

mach_wind       = wind_solution/ sound_speed(u = wind_solution, x= x_array, gamma= gamma_val)
mach_accretion  = accretion_solution/ sound_speed(u = accretion_solution, x= x_array, gamma= gamma_val)
```



**Isothermal case ($\gamma = 1$)**

Let us revisit Eq. (14). As we take $\lim \gamma \to 1$ ; $1/(\gamma - 1)$ term blows up. Let us group that terms together.

$$\frac{u^2}{2} + \frac{u^{1-\gamma} x^{-2(\gamma-1)}}{\gamma - 1} = \left[\frac{5 - 3\gamma}{2(\gamma - 1)}\right] + \frac{2}{x}$$

$$\frac{u^2}{2} + \frac{u^{1-\gamma} x^{-2(\gamma-1)} - 1}{\gamma - 1} = \frac{5 - 3\gamma - 2}{2(\gamma - 1)} + \frac{2}{x}$$

$$= \frac{3(1-\gamma)}{2(\gamma-1)} + \frac{2}{x} = -\frac{3}{2} + \frac{2}{x}$$

Now we apply the famous limit identity,

$$\boxed{\lim_{\varepsilon \to 0} \frac{a^\varepsilon - 1}{\varepsilon} = \ln a}$$

Set $\varepsilon = 1 - \gamma$ and $a \equiv ux^2$

$$\lim_{\gamma \to 1} \frac{u^{1-\gamma} x^{-2(\gamma-1)} - 1}{\gamma - 1} = -\ln\left(ux^2\right) = -\ln u - 2\ln x$$

Hence, rearranging the above equation we get the famous **isothermal** Parker wind solution.

$$\frac{u^2}{2} - \ln u = 2\ln x + \frac{2}{x} - \frac{3}{2} \tag{15}$$

$$\left(\frac{v}{a_c}\right)^2 - 2\ln\left(\frac{v}{a_c}\right) = 4\ln\frac{r}{r_c} + \frac{2GM}{r a_c^2} \tag{16}$$

But, it must be noted that, unlike the adiabatic case, the sound speed $a_c = a(r) = \sqrt{RT}$ is a *constant* of motion.

Qualitatively an isothermal equation of state ($\gamma = 1$) implies an idealised, infinitely efficient heating/cooling mechanism. For Parker wind, it means energy is continuously injected into gas as it expands to keep it hot ($T = \text{const}$). For Bondi accretion, it means compressional heating is perfectly radiated away as the gas falls in – unlike adiabatic case where heat is NOT exchanged with the surrounding.

## EXTRA : Determining Non-Transonic solutions

Let us get back to Eq. (10)

$$\frac{\mathrm{d}v}{\mathrm{d}r} = \frac{\frac{-GM}{vr^2}\left(1 - \frac{2ra^2}{GM}\right)}{\left(1 - \frac{a^2}{v^2}\right)}$$

Since in these curves, we no longer have the sonic points $(r_c, v_c)$, we have **two** independent parameters which determine the curves. They are

$$\varepsilon = \frac{v(r)^2}{2} + \frac{a(r)^2}{\gamma - 1} - \frac{GM}{r^2} = \text{constant}$$

and

$$a^{2/(\gamma-1)} v r^2 = \text{constant}$$

We can undimensionalise both of these equations, and solve for a common root $u(x)$ for each $x$, but that will be cumbersome. Instead, we want to directly get the curve by integrating Eq. (10) from a given starting point using RK4. Let's convert this into our convenient dimensionless form.

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{-2}{ux^2}\left(\frac{a_c^2 - xa(x)^2}{a_c^2 - [a(x)/u(x)]^2}\right) \tag{17}$$

subject to the constraint

$$a(x)^{2/(\gamma-1)} u x^2 = a(x_0)^{2/(\gamma-1)} u_0 x_0^2 \tag{18}$$

along a curve which starts from the point $(u_0, x_0)$. We take 6 such points starting from $x_0$ (3 below *subsonic* wind solution, and 3 above *supersonic* accretion solution), with arbitrarily chosen $u_0 \equiv u(x_0)$.

```
down0x = np.ones(3)*(df['x (r/r_c)'][0])
# arrays of x0 's
down0u = np.array([0.12 , df['Wind u(x)'][0]/10 , df['Wind u(x)'][0]/100])
# arrays of u0 's

up0x = down0x
up0u = np.array([18.5 , df['Accretion u(x)'][0] * 10 , df['Accretion u(x)'][0] *
    100])
```

In Eq. (17), we set $\boxed{a_c = 1}$ for simplicity. And $a(x)$ is determined through the following relation

$$a(x) = a(x_0)\left(\frac{u_0 x_0^2}{u(x)x^2}\right)^{(\gamma-1)/2}$$

A constant value of $\boxed{a(x_0) = 9}$ is taken for all the curves.

```
def bondi_ode(x, u, C, gamma):
    """Defines du/dx given the constraint-substituted sound speed."""
    # Prevent division by zero or negative bases due to numerical fluctuations
    if u <= 0 or x <= 0:
        return 0

    # Calculate local sound speed squared from the constraint
    a2x = (C / (u * x**2))**(gamma - 1)

    # a_c = 1 (for simplicity)
    numerator = 1.0 - x * a2x
    denominator = 1.0 - (a2x / u**2)

    return (-2.0 / (u * x**2)) * (numerator / denominator)
```

```python
def rk4_integrate(x_span, u0, C, gamma):
    """Performs RK4 integration forward in x."""
    u_vals = np.zeros(len(x_span))
    u_vals[0] = u0

    for i in range(len(x_span) - 1):
        x = x_span[i]
        u = u_vals[i]
        dx = x_span[i+1] - x_span[i]

        k1 = bondi_ode(x, u, C, gamma)
        k2 = bondi_ode(x + dx/2, u + dx*k1/2, C, gamma)
        k3 = bondi_ode(x + dx/2, u + dx*k2/2, C, gamma)
        k4 = bondi_ode(x + dx, u + dx*k3, C, gamma)

        u_vals[i+1] = u + (dx / 6.0) * (k1 + 2*k2 + 2*k3 + k4)

    return u_vals
```

Integrating,

```python
# Grid for integration matching the table array length/span
x_integration = np.logspace(-2, 2, 100)

# Dictionary to hold data for the output table
table_output = {'x': x_integration}

# --- Integrate the bottom 3 curves --- SUBSONIC
curves_data1 = []
for i, u0 in enumerate(down0u):
    # C constant evaluated at initial conditions where a2_0 matches the critical background
    C_val = a0x**(2/(gamma_val -1)) * u0 * down0x[i]**2

    # Integrate using RK4
    u_sol = rk4_integrate(x_integration, u0, C_val, gamma_val)
    curves_data1.append(u_sol)
    table_output[f'Curve_{i+1}_u'] = u_sol

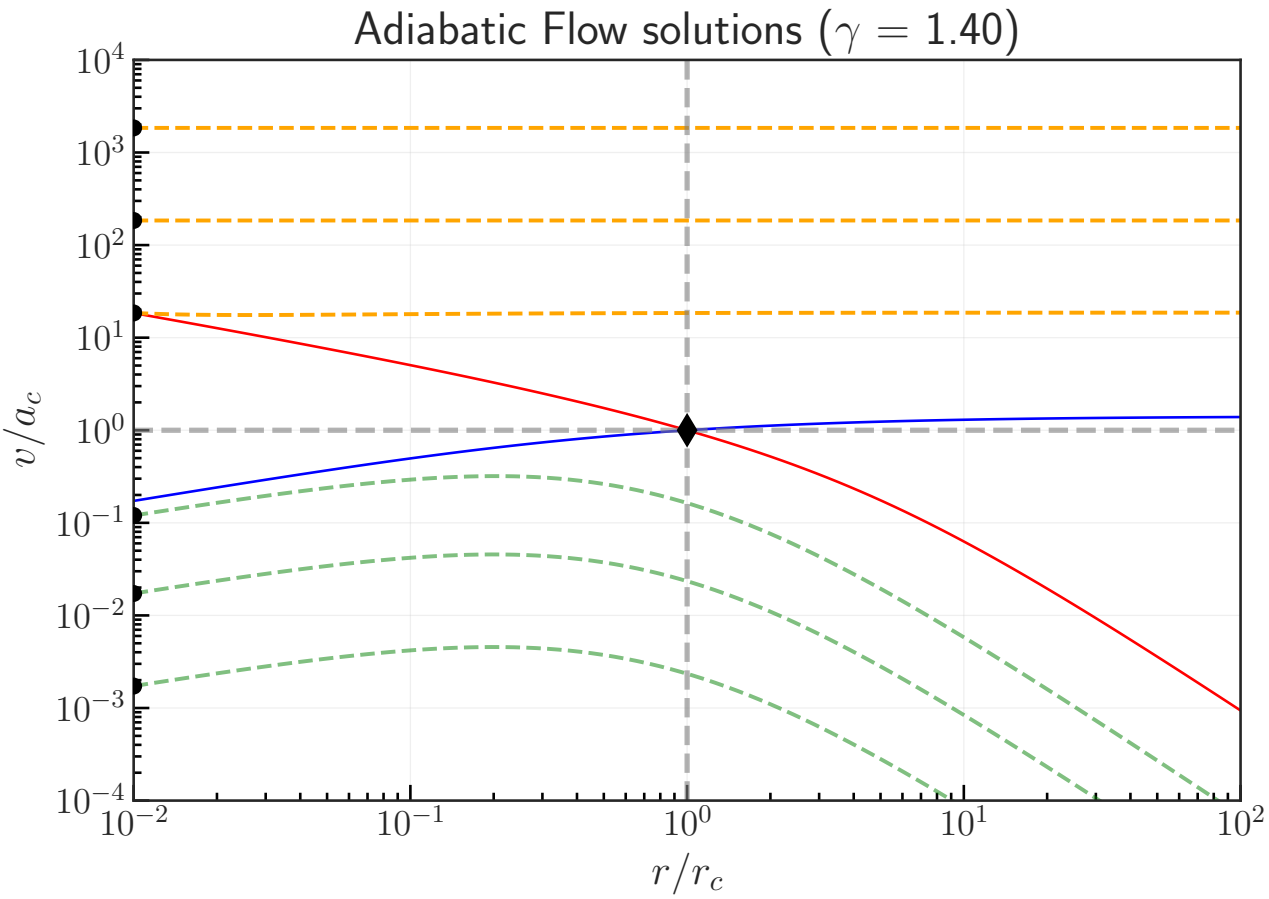
# ---  Integrate the top 3 curves --- SUPERSONIC
curves_data2 = []
for i, u0 in enumerate(up0u):
    # C constant evaluated at initial conditions where a2_0 matches the critical background
    C_val = a0x**(2/(gamma_val -1)) * u0 * up0x[i]**2

    # Integrate using RK4
    u_sol = rk4_integrate(x_integration, u0, C_val, gamma_val)
    curves_data2.append(u_sol)
    table_output[f'Curve_{i+1}_u'] = u_sol
```

Overlaying them,

```python
for i, u_sol in enumerate(curves_data1):
    # SUBSONIC SOLUTIONS
    plt.plot(x_integration, u_sol, color='green', alpha = 0.5, linewidth=1.5, ls= '--')
    # Mark their starting points
    plt.scatter(down0x[i], down0u[i], color='k', s=30, zorder=5)

for i, u_sol in enumerate(curves_data2):
    # SUPERSONIC SOLUTIONS
    plt.plot(x_integration, u_sol, color='orange', alpha =1, linewidth=1.5, ls= '--')
    # Mark their starting points
    plt.scatter(up0x[i], up0u[i], color='k', s=30, zorder=5)
```




This is the final combined figure for $\gamma$ = 1.40. Note that we have not plotted the physically prohibited solutions arising on either side of the sonic point for the sake of brevity.

## References


- https://www.astronomy.ohio-state.edu/ryden.1/ast825/ch8.pdf



Sincere thanks to Prof. Sudip Kr. Garain for kindly reviewing my solutions.