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Subj: PH4212 **ASSIGNMENT**

Question 3).

(A) Define a 2-level system

E_2 (or E_e)

E_1 or ground state. (E_g)

~~Some~~ variables

Rabi frequency $= \Omega$

Detuning frequency, $\Delta = \omega - \omega_0$.

$$\omega_0 = \frac{E_2 - E_1}{\hbar} = \frac{\hbar\omega_2 - \hbar\omega_1}{\hbar} = \omega_2 - \omega_1$$

Spontaneous decay rate, $\Gamma = A_{21}$

Ans (a) Interaction of atoms with radiation is studied in a semi-classical approach.

we start from $i\hbar \frac{\partial \Psi}{\partial t} = H\Psi$ ... (*1)

where, $\hat{H} = \hat{H}_0 + \hat{\mu} E(t)$

Total Hamiltonian = Reference + time-dependent interaction of molecular dipole with an oscillating EM field.

$$E(t) = 2E_0 \cos(\omega t) = E_0 \left[e^{i\omega t} + e^{-i\omega t}\right]$$

Then we can use eigenstates basis of $\hat{H}_0$ to write

$$\hat{H} = \begin{pmatrix} \hbar\omega_1 & \mu E(t) \\ \mu E(t) & \hbar\omega_2 \end{pmatrix}$$

For (*1), we can write the total wavefunction as a superposition of the stationary states of H_0, i.e

$$\Psi(\vec{r},t) = c_1(t)\,\psi_1(\vec{r})\,e^{-iE_1t/\hbar} + c_2(t)\,\psi_2(\vec{r})\,e^{-iE_2t/\hbar}$$

- Writing in new notation)

$$\Psi(\vec{r},t) = c_1|1\rangle e^{-i\omega_1 t} + c_2|2\rangle e^{-i\omega_2 t}.$$

$$|c_1|^2 + |c_2|^2 = 1 \quad \forall t.$$

Density Matrix

$$\rho = |\Psi\rangle\langle\Psi| = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} = \begin{pmatrix} |c_1|^2 & c_1 c_2^* \\ c_2 c_1^* & |c_2|^2 \end{pmatrix}$$

Interpretation

$|c_1|^2, |c_2|^2 \rightarrow$ population at each level.

off diagonal $\rightarrow$ coherences $\rightarrow$ represent the response of the system at the driving frequency.

The time dependence of density matrix can be described using Liouville Von-Neumann Equation —

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[\hat{H}, \rho].$$

Liouville Representation $\rho \equiv \begin{pmatrix} \rho_{11} \\ \rho_{12} \\ \rho_{21} \\ \rho_{22} \end{pmatrix}$ s.t $\frac{d\rho}{dt} = \frac{-i}{\hbar}\hat{L}\rho.$

$$\frac{d\rho}{dt} = \frac{-i}{\hbar}\left[\begin{pmatrix} \hbar\omega_1 & \mu E(t) \\ \mu E(t) & \hbar\omega_2 \end{pmatrix} * \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}\right.$$

$$\left. - \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}\begin{pmatrix} \hbar\omega_1 & \mu E(t) \\ \mu E(t) & \hbar\omega_2 \end{pmatrix}\right]$$

$$= \frac{-i}{\hbar}\begin{bmatrix} \hbar\omega_1\rho_{11} + \mu E(t)\rho_{21} & \hbar\omega_1\rho_{12} + \rho_{22}\mu E(t) \\ \rho_{11}\mu E(t) + \rho_{21}\hbar\omega_2 & \mu E(t)\rho_{12} + \rho_{22}\hbar\omega_2 \end{bmatrix}$$

$$* - \begin{bmatrix} \rho_{11}\hbar\omega_1 + \rho_{12}\mu E(t) & \rho_{11}\mu E(t) + \rho_{12}\hbar\omega_2 \\ \rho_{21}\hbar\omega_1 + \rho_{22}\mu E(t) & \rho_{21}\mu E(t) + \rho_{22}\hbar\omega_2 \end{bmatrix}$$

$$\therefore \frac{d\rho_{11}}{dt} = \frac{-i}{\hbar}\left[\hbar\omega_1\rho_{11} + \mu E(t)\rho_{21} - \rho_{11}\hbar\omega_1 - \rho_{12}\mu E(t)\right]$$

$$\frac{d\rho_{12}}{dt} = -\frac{i}{\hbar}\left[\hbar\omega_1\rho_{12} + \rho_{22}\mu E(t) - \rho_{11}\mu E(t) - \rho_{12}\hbar\omega_2\right]$$

$$= -\frac{i}{\hbar}\left[\rho_{12}\left(\hbar(\omega_1-\omega_2)\right) + \rho_{22}\mu E - \rho_{11}\mu E\right]$$

$$\frac{d\rho_{21}}{dt} = -\frac{i}{\hbar}\left[\rho_{11}\mu E + \rho_{21}\hbar\omega_2 - \rho_{21}\hbar\omega_1 - \rho_{22}\hbar\omega_1\right]$$

$$= -\frac{i}{\hbar}\left[\rho_{11}\mu E + \rho_{21}\left(\hbar(\omega_2-\omega_1)\right) - \rho_{22}\hbar\omega_1\right]$$

$$\frac{d\rho_{22}}{dt} = -\frac{i}{\hbar}\left[\mu E\rho_{12} + \rho_{22}\hbar\omega_2 - \rho_{21}\mu E - \rho_{22}\hbar\omega_2\right]$$

with Liouville representation, we can write a matrix compact notation.

$$\frac{d}{dt}\begin{pmatrix}\rho_{11}\\ \rho_{12}\\ \rho_{21}\\ \rho_{22}\end{pmatrix} = -\frac{i}{\hbar}\begin{bmatrix} 0 & -\mu E(t) & \mu E(t) & 0 \\ -\mu E(t) & -\hbar\omega_0 & 0 & \mu E(t) \\ +\mu E(t) & 0 & +\hbar\omega_0 & -\mu E \\ 0 & +\mu E & -\mu E & 0 \end{bmatrix}\begin{pmatrix}\rho_{11}\\ \rho_{12}\\ \rho_{21}\\ \rho_{22}\end{pmatrix}$$

Transformation to Rotating frame :-

In the lab frame, the coherences ρ_{12}, ρ_{21} oscillate at the high optical frequency ω_0. To simplify, we transform to a frame rotating at driving frequency 'ω'. We define slowly varying components

$\tilde{\rho}_{ij}$

Define new density matrix $\tilde{\rho}(t)$

$$\tilde{\rho}_{11}(t) = \rho_{11}(t)$$
$$\tilde{\rho}_{22}(t) = \rho_{22}(t)$$

} populations will remain unaffected.

$$\tilde{\rho}_{21}(t) = \rho_{21} e^{i\omega t}$$
$$\tilde{\rho}_{12}(t) = \rho_{12} e^{-i\omega t}$$

$\omega \equiv$ frequency of oscillating field, which will be near resonance for frequency ω_0.

By defining $\tilde{\rho}(t)$, we are separating out the rapidly oscillating part of the molecular response to the field, and keeping only some slowly varying part of the envelope.

$$\therefore \frac{d\tilde{\rho}_{11}}{dt} = -\frac{i}{\hbar}\left[\mu E(t)\,\rho_{21} - \rho_{12}\,\mu E(t)\right]$$

$$= -\frac{i}{\hbar}\left[\mu E(t)\,\tilde{\rho}_{21} e^{-i\omega t} - \tilde{\rho}_{12} e^{i\omega t}\,\mu E(t)\right]$$

Define: $\Omega = \frac{\mu E_0}{\hbar}$ (R) $\tilde{\Omega}(t) \equiv \frac{\mu E(t)}{\hbar} e^{i\omega t}$ (C)

$$\tilde{\Omega}(t) = \frac{\mu}{\hbar} E_0 \times 2\cos\omega t\, e^{i\omega t}$$

$$= \underbrace{2\Omega\cos\omega t}_{R}\, e^{i\omega t}$$

$$\tilde{\Omega}^*(t) = 2\Omega\cos\omega t\, e^{-i\omega t}$$

$$\therefore \frac{d\tilde{\rho}_{11}}{dt} = -i\left(\tilde{\Omega}^*(t)\,\tilde{\rho}_{21} - \tilde{\Omega}(t)\,\tilde{\rho}_{12}\right)$$

Similarly $\frac{d\tilde{\rho}_{22}}{dt} = -\frac{i}{\hbar}\left[\mu E(t)\,\rho_{12} - \rho_{21}\,\mu E(t)\right]$

$$= -i\left(\tilde{\Omega}(t)\,\tilde{\rho}_{12} - \tilde{\Omega}^*(t)\,\tilde{\rho}_{21}\right)$$

Now,

$$\frac{d\rho_{12}}{dt} = -\frac{i}{\hbar}\left[-\mu E(t)\rho_{11} - \hbar\omega_0\rho_{12} + \mu E(t)\rho_{22}\right]$$

$$= -\frac{i}{\hbar}\left[-\mu E\tilde{\rho}_{11} - \hbar\omega_0 e^{i\omega t}\tilde{\rho}_{12} + \mu E\tilde{\rho}_{22}\right]$$

And,

$$\frac{d\rho_{21}}{dt} = -\frac{i}{\hbar}\left[\mu E\rho_{11} + \hbar\omega_0\rho_{21} - \mu E\rho_{22}\right].$$

$$= -\frac{i}{\hbar}\left[\mu E\tilde{\rho}_{11} + \hbar\omega_0 e^{-i\omega t}\tilde{\rho}_{21} - \mu E\tilde{\rho}_{22}\right].$$

$$\therefore \frac{d\tilde{\rho}_{12}}{dt} = \frac{d}{dt}\left[\rho_{12}e^{-i\omega t}\right]$$

$$= \frac{d}{dt}[\rho_{12}]e^{-i\omega t} + \rho_{12}(-i\omega)e^{-i\omega t}$$

$$= \frac{d}{dt}[\rho_{12}]e^{-i\omega t} - (i\omega)\tilde{\rho}_{12}$$

$$= i\left(+\tilde{\Omega}^*(t)\tilde{\rho}_{11} - \tilde{\Omega}^*(t)\tilde{\rho}_{22}\right) + i\omega_0 e^{i\omega t}\tilde{\rho}_{12} - i\omega\tilde{\rho}_{12}$$

Define:- $-\omega_0 + \omega \equiv \Delta$

Detuning parameter.

$$= -i\left(-\Delta\tilde{\rho}_{12} - \tilde{\Omega}^*\tilde{\rho}_{11} + \tilde{\Omega}^*\tilde{\rho}_{22}\right)$$

similarly

$$\frac{d\tilde{\rho}_{21}}{dt} = -i\left[\tilde{\Omega}(t)\tilde{\rho}_{11} - \Delta\tilde{\rho}_{21} - \tilde{\Omega}(t)\tilde{\rho}_{22}\right]$$

Rotating wave approximation

$$\tilde{\Omega}(t) = \frac{\mu E(t)}{\hbar} e^{i\omega t}$$

$$= \frac{\mu E_0}{\hbar} 2\cos\omega t \, e^{i\omega t}$$

$$= \frac{\mu E_0}{\hbar} \left[e^{i\omega t} + e^{-i\omega t}\right] e^{i\omega t}$$

$$= \frac{\mu E_0}{\hbar} \left[e^{2i\omega t} + 1\right]$$

$$= \Omega \left[e^{2i\omega t} + 1\right]$$

In most case $|\Delta| = |\omega - \omega_0| \ll \omega_0$.

upon integrating $\int e^{2i\omega t} dt \simeq 0$ (as it is quickly oscillating. so, it will have no effect and.

we can make approximation, $\boxed{\tilde{\Omega}(t) \simeq \Omega}$.

This is valid where the electric field is weak enough that the Rabi frequency Ω is much slower than ω.

∴ OBEs

$$\frac{d}{dt}\begin{pmatrix} \tilde{\rho}_{11} \\ \tilde{\rho}_{12} \\ \tilde{\rho}_{21} \\ \tilde{\rho}_{22} \end{pmatrix} = -i \begin{bmatrix} 0 & -\Omega & \Omega & 0 \\ -\Omega & \Delta & 0 & \Omega \\ \Omega & 0 & -\Delta & -\Omega \\ 0 & \Omega & -\Omega & 0 \end{bmatrix} \begin{bmatrix} \tilde{\rho}_{11} \\ \tilde{\rho}_{12} \\ \tilde{\rho}_{21} \\ \tilde{\rho}_{22} \end{bmatrix}$$

New definitions

$$u \equiv \tilde{\rho}_{12} + \tilde{\rho}_{21} = 2\,\mathrm{Re}(\tilde{\rho}_{12})$$

$$v = -i(\tilde{\rho}_{12} - \tilde{\rho}_{21}) = 2\,\mathrm{Im}(\tilde{\rho}_{12})$$

$$w = \rho_{22} - \rho_{11}$$

u, v are the inphase and quadrature components of dipole in a frame rotating at ω.

$$\dot{u} = \frac{d}{dt}\tilde{\rho}_{12} + \frac{d\tilde{\rho}_{21}}{dt}$$

$$= -i\left[-\Omega\tilde{\rho}_{11} + \Delta\tilde{\rho}_{12} + \Omega\tilde{\rho}_{22}\right] - i\left[\Omega\tilde{\rho}_{11} - \Delta\tilde{\rho}_{21} - \Omega\tilde{\rho}_{22}\right]$$

$$= -i\,\Delta\left[\tilde{\rho}_{12} - \tilde{\rho}_{21}\right] = \Delta v$$

Similarly,

$$\dot{v} = -i\left[\frac{d\tilde{\rho}_{12}}{dt} - \frac{d\tilde{\rho}_{21}}{dt}\right] = -1\left[-\Omega\tilde{\rho}_{11} + \Delta\tilde{\rho}_{12} + \Omega\tilde{\rho}_{22} - \Omega\tilde{\rho}_{11} + \Delta\tilde{\rho}_{21} + \Omega\tilde{\rho}_{22}\right]$$

$$= \Omega\left[\tilde{\rho}_{11} - \tilde{\rho}_{22} + \tilde{\rho}_{11} - \tilde{\rho}_{22}\right] + \Delta\left[-\tilde{\rho}_{12} - \tilde{\rho}_{21}\right]$$

$$\dot{v} = -2w\Omega - \Delta u$$

And $w = \rho_{22} - \rho_{11} = \tilde{\rho}_{22} - \tilde{\rho}_{11}$ $\therefore$ $\dot{w} = \frac{d\tilde{\rho}_{22}}{dt} - \frac{d\tilde{\rho}_{11}}{dt}$

$$= -i\left[\Omega\tilde{\rho}_{12} - \Omega\tilde{\rho}_{21} + \Omega\tilde{\rho}_{12} - \Omega\tilde{\rho}_{21}\right]$$

$$= -i \times 2\Omega\left[\tilde{\rho}_{12} - \tilde{\rho}_{21}\right]$$

$$= 2\Omega v$$

∴ New OBEs

$$\dot{u} = \Delta v$$

$$\dot{v} = -2w\Omega - \Delta u$$

$$\dot{w} = 2\Omega v$$

Inclusion of the Interaction Rate.

$\Gamma \equiv A_{21}$

If atoms decay from state $|2\rangle$ to $|1\rangle$ at rate $-\Gamma$

$$\dot{\tilde{\rho}}_{22} \to \dot{\tilde{\rho}}_{22} - \Gamma\tilde{\rho}_{22} \quad ; \quad \dot{\tilde{\rho}}_{11} \to \dot{\tilde{\rho}}_{11} + \Gamma\tilde{\rho}_{22}$$

gain.

while off-diagonal elements will decay at rate.

$r = \Gamma/2$

$$\dot{\tilde{\rho}}_{12} \to \dot{\tilde{\rho}}_{12} - \frac{\Gamma}{2}\tilde{\rho}_{12} \quad ; \quad \dot{\tilde{\rho}}_{21} \to \dot{\tilde{\rho}}_{21} - \frac{\Gamma}{2}\tilde{\rho}_{21}$$

$$\dot{u} = \frac{d}{dt}\tilde{\rho}_{12} + \frac{d}{dt}\tilde{\rho}_{21} - \frac{\Gamma}{2}(\tilde{\rho}_{12} + \tilde{\rho}_{21}) = \Delta v + u\left(-\frac{\Gamma}{2}\right)$$

$$\dot{w} = 2\Omega v - \Gamma \tilde{\rho}_{22} \times 2$$

$$= 2\Omega v - \Gamma(1+w)$$

Now,

$$\rho_{22} + \rho_{11} = w$$

$$\rho_{22} + \rho_{11} = 1$$

$$\rho_{22} = \frac{1+w}{2}$$

similarly,

$$\dot{v} = -\Delta u - 2\Omega w - \frac{\Gamma}{2} v$$

$$\dot{u} = \Delta v - \frac{\Gamma}{2} u$$

$$\dot{v} = -\Delta u - 2w\Omega - \frac{\Gamma}{2} v$$

$$\dot{w} = 2\Omega v - \Gamma(1+w)$$

Ans.

These equations describe the motion of the Bloch vector on or inside the Bloch sphere, where Γ terms drive the vector toward the equilibrium state (the south pole, $w = -1$).

Physical Importance

- **v (Absorptive component)** — It represents the part of the induced atomic dipole moment oscillating 90° out-of-phase (in quadrature) with the field, which physically determine the rate of energy exchange b/w the atom and the laser.
- **u (Dispersive component)** — It represents the part of the induced atomic dipole moment oscillating in-phase with the driving electric field, which physically determines the refractive index and the dispersive shift of the light.

3 (b) Find Steady state solutions.

$\dot{u} = 0$
$\dot{w} = 0$
$\dot{v} = 0$

It turns out, The Rabi frequency in the question is $2\times$ ~~Ω~~. (at my ams) ∴ we define.

$\Omega' \equiv 2\Omega$

- $\Delta v = \frac{\Gamma}{2} u \rightarrow u = \frac{2\Delta}{\Gamma} v$

- $\dot{u} = -\Delta v - \frac{\Gamma}{2} u + \Omega w = 0$ ... (11)

- $\dot{v} = \Omega' u - \Gamma v = 0$

$\dot{w} = \Gamma(1+w) \rightarrow w = \frac{\Omega' v}{\Gamma} - 1$

Solving For Bloch. components.

subst. in eq (11)

$$\frac{2\Delta^2}{\Gamma} v + \Omega' \left[\frac{\Omega'}{\Gamma} v - 1\right] = -\frac{\Gamma}{2} v$$

$$2\Delta^2 v + \Omega'[\Omega' v - \Gamma] = -\frac{\Gamma^2}{2} v$$

$$4\Delta^2 v + 2\Omega'[\Omega' v - \Gamma] = -\Gamma^2 v$$

$$4\Delta^2 v + 2\Omega'^2 v - 2\Omega'\Gamma = -\Gamma^2 v$$

$$v[4\Delta^2 + 2\Omega'^2 + \Gamma^2] = 2\Omega'\Gamma$$

$$v_{ss} = \frac{2\Omega'\Gamma}{4\Delta^2 + 2\Omega'^2 + \Gamma^2} = \frac{\Omega'\Gamma/2}{\Delta^2 + \Gamma^2/4 + \Omega'^2/2}$$

Excited-State Population

Recall, $\rho_{22}^{ss} = \frac{1}{2}(w_{ss} + 1) = \frac{1}{2}\frac{2\Omega' v_{ss}}{\Gamma}$

$$= \frac{\Omega'^2\Gamma/4}{\Delta^2 + \Gamma^2/4 + \Omega'^2/2} = \frac{2\Omega}{\Gamma}\cdot\frac{4\Omega'}{4\Delta^2 + 8\Omega^2 + \Gamma^2} = \frac{8\Omega^2}{4\Delta^2 + 8\Omega^2 + \Gamma^2}$$

proved

Define

on Resonance saturation parameter, $s_0 = \frac{2\Omega'^2}{\Gamma^2}$.

Rewriting $\rho_{22}^{ss} = \frac{s_0/2}{1 + s_0 + \left(\frac{2\Delta}{\Gamma}\right)^2}$.

At Resonance

$$\rho_{22}^{ss}(\Delta = 0) = \frac{1}{2}\left[\frac{s_0}{1+s_0}\right] \xrightarrow{s_0 \to \infty} \frac{1}{2}$$

proved.

Laser Intensity is very high

∴ Laser can almost equalize the population of two levels.

③ ⓒ steady state solutions are:

$$v_{ss} = \frac{\Omega'\Gamma/2}{\Delta^2 + \Gamma^2/4 + \Omega'^2/2} \; ; \; u_{ss} = \frac{\Omega'\Delta}{\Delta^2 + \Gamma^2/4 + \Omega'^2/2}$$

$$w_{ss} = \frac{-(\Delta^2 + \Gamma^2/4)}{\Delta^2 + \Gamma^2/4 + \Omega'^2/2}$$

③ ⓒ From steady state solutions, write the line-shapes $\chi''(\Delta)$ - absorption, $\chi'(\Delta)$ - dispersion as functions of detuning.

The atomic susceptibility $\chi(\Delta) = \chi' + i\chi''$ describes how a medium responds to an electric field.

Recall, $\chi''(\Delta)$ - absorption $\propto v_{ss}$.

$\chi'(\Delta)$ - dispersion $\propto u_{ss}$.

$\therefore$ ~~$\chi'(\Delta) \propto \Omega\Delta$~~

▷ $\chi'(\Delta) \propto \dfrac{\Omega'\Delta}{\Delta^2 + \Gamma^2/4 + \Omega'^2/2}$

▷ $\chi''(\Delta) \propto \dfrac{\Omega'\Gamma/2}{\Delta^2 + \Gamma^2/4 + \Omega'^2/2}$

Weak driving limit ($\Omega' \to 0$)

When laser intensity is very low,

$\dfrac{\Omega'^2}{2} \ll \Delta^2 + \Gamma^2/4$ $\therefore$ can be ignored

The absorption profile, becomes

$\therefore \chi''(\Delta) \sim \dfrac{\Omega'\Gamma/2}{\Delta^2 + \Gamma^2/4}$.

Hence, we have the classic Lorentzian Profile.

$\therefore$ The HWHM, $\gamma_{weak} = \Gamma/2$. Proved

ong Driving ($\Omega >> 1$). Find Effective linewidth.

$$\chi''(\Delta) \propto \frac{\Gamma\Omega'/2}{\left(\frac{\Gamma^2}{4} + \frac{\Omega^2}{2}\right) + \Delta^2} \equiv \frac{\Gamma\Omega'/2}{\left(\frac{\Gamma_{eff}}{2}\right)^2 + \Delta^2}$$

$$\therefore \frac{\Gamma_{eff}}{2} = \frac{1}{2}\sqrt{\Gamma^2 + 2\Omega^2} \Rightarrow \boxed{\Gamma_{eff} = \sqrt{\Gamma^2 + 2\Omega'^2}}$$

Power Broadening

- Clearly in absence of driving, $s_0 = 2\Omega'^2/\Gamma^2 = 0$

$\therefore \Gamma_{eff} = \Gamma$

New linewidth, $\Gamma_{eff} = \sqrt{\Gamma^2 + s_0\Gamma^2} = \Gamma\sqrt{1 + s_0}$

$s_0 = 3 \rightarrow \Gamma_{eff} = 2\Gamma$

$\%$ Increase $\Rightarrow \dfrac{2\Gamma - \Gamma}{\Gamma} \times 100 = \boxed{100\%}$

(d) In the dressed state picture, eigenstates of the combined atom-field system have energies $\pm \frac{\hbar}{2}\sqrt{\Delta^2+\Omega^2}$, and ground-state energy acquires an A.C stark shift:

$$\delta E_g \simeq \frac{\hbar\Omega^2}{4\Delta} \quad |\Delta| \gg \Omega.$$

using this, and steady state u_{ss}, show that light shift is consistent with dispersive component of the atomic polarisation (i.e u')

Ans For $|\Delta| \gg \Omega'$.

Dressed state energies are, $\pm\frac{\hbar}{2}\sqrt{\Delta^2+\Omega^2}$

$$= \pm\frac{\hbar}{2}|\Delta|\sqrt{1+\frac{\Omega^2}{\Delta^2}} \simeq \pm\frac{\hbar}{2}\left(|\Delta| \pm \frac{\Omega'^2}{2|\Delta|}\right)$$

$$\therefore \delta E_g = \pm\frac{\Omega'^2}{4|\Delta|}$$

Now, steady-state dispersive component is :-

$$u_{ss} = \frac{\Omega'\Delta}{\Delta^2+\Gamma^2/4+\Omega'^2/2}$$

For $\Delta \gg \Gamma, \Omega$, $\therefore u_{ss} \simeq \frac{\Omega'}{\Delta}$

Now, $\Omega = \frac{dE}{\hbar}$ or where $\langle d\rangle \propto -\frac{d\Omega}{D}$ induced atomic dipole moment.

$$\Rightarrow \langle d\rangle \propto -\frac{d^2E}{\hbar\Delta} \quad \therefore \text{Polarizability } \alpha \propto \frac{d^2}{\hbar\Delta}$$

$\therefore$ Interaction Energy, $u' = -\frac{1}{2}\alpha E^2$

$$u \propto \frac{d^2E^2}{2\hbar\Delta} \propto \frac{\hbar\Omega^2}{2\Delta}$$

$\therefore \delta E_g \simeq \frac{\hbar\Omega^2}{4\Delta}$ ($\therefore$ Yes it is consistent.

Estimate δE_g (in Hz)

For Na (Sodium) $\lambda = 589$ nm, $\Gamma = 2\pi \times 9.8$ MHz.

$|\langle 2|er|1\rangle| = 3.5\, ea_0$.

driven 1 GHz off resonance at intensity

$I = 10\ mW\,cm^{-2}$.

For, $a_0 = 5.29 \times 10^{-11}$ m.

$\therefore |\langle 2|er|1\rangle| = 3.5\, e a_0 = 2.96 \times 10^{-29}$ C.m.

$\Delta = 2\pi \times 1$ GHz.

$I = 10\ mW cm^{-2} = 100\ W m^{-2}$

Now, For electric field, we know,

$$I = \frac{1}{2} c \epsilon_0 |E_0|^2 \rightarrow |E_0| = \sqrt{\frac{2I}{c\epsilon_0}}$$

$$= \sqrt{\frac{200}{(3\times10^8)(8.85\times10^{-12})}}$$

$$\simeq 274.4\ V m^{-1}.$$

$\therefore$ Rabi Frequency, $\Omega = \frac{d|E_0|}{\hbar} = \frac{(2.96\times10^{-29}) \times 274.4}{1.054\times10^{-34}}$

$$\simeq 7.72 \times 10^7\ rad/s.$$

Now, we calculate the light shift in (Hz)

$$\delta\nu = \frac{\delta E_g}{h} = \frac{1}{2\pi}\frac{\Omega^2}{4\Delta} =$$

$$= \frac{(7.72\times10^7\ s^{-1})^2}{2\pi \times 4 \times 4\pi^2 (10^9)\ s^{-1}}$$

$$\simeq 37.7\ KHz$$

Modern atomic clock have resonance linewidths of the order of 1 Hz. $\therefore$ Yes, this shift is detectable in atomic clocks.

Question 5 — Multi Electron Atoms, From LS to jj

(a) For two equivalent p- electrons (np^2), enumerate all possible two-electron states using Pauli exclusion principle and LS coupling.

Using Hund's rules, identify the ground term.

Ans For p- electron, $l = 1$.

2 equivalent electrons ⇒ $l_1 = 1$, $s_1 = 1/2$

$l_2 = 1$, $s_2 = 1/2$.

In LS coupling, we first combine individual orbital momenta into L, and individual spins into S.

$L = |l_1 - l_2|, \ldots, l_1 + l_2 \;|\; L = |1-1|, \ldots 1+1 = 0, 1, 2$. [S, P, and D]

$S = |s_1 - s_2|, \ldots, s_1 + s_2$.

$S = \left|\frac{1}{2} - \frac{1}{2}\right|, \ldots \frac{1}{2} + \frac{1}{2} = 0, 1$

- $S = 0$ is a Singlet (Antisymmetric)
- $S = 1$ is Triplet (Symmetric).

The Pauli exclusion principle forbids both e⁻ occupying the same single particle state, and antisymmetrisation identifies $|a, b\rangle$ with $-|b, a\rangle$.

(L,S) terms before applying Pauli Exclusion Principle.

Two electrons with $l_1 = l_2 = 1$ couple to 3 L states

2 2S states

This gives 6 terms — $^1S, ^3S, ^1P, ^3P, ^1D, ^3D$.

Total: $\sum (2L+1)(2S+1) = (1+3) + (3+9) + (5+15) = 36$ states.

count for "non"-equivalent electron.

Now, $\Psi_{total} = \psi_{space} \times \chi_{spin}$ → must be antisymmetric.

To ensure Ψ_{total} is antisymmetric, we must pair a symmetric spatial state with an antisymmetric spin state, or vice versa.

ψ_{space} → Symmetric → $(L+2l)$ even $\xrightarrow{l=1}$ L is even

ψ_{space} → Anti-sym → $(L+2l)$ odd $\xrightarrow{l=1}$ L is odd

χ_{spin} → Sym → $S=1$; Asym → $S=0$] $(-1)^{S+1}$ parity.

$\therefore (-1)^L \cdot (-1)^{S+1} = -1 \iff L+S = \text{even}$.

Term	L	S	L+S		Multiplicity = (2L+1)(2S+1)
^1S	0	0	0	✓	1
^3S	0	1	1	✗	
^1P	1	0	1	✗	
^3P	1	1	2	✓	9
^1D	2	0	2	✓	5
^3D	2	1	3	✗	

∴ Total allowed states = 1 + 9 + 5 = 15

The 3 Allowed terms are $^1S, ^3P, ^1D$

Spectroscopic Notation: $^{2S+1}L_J$

$J = |L-S|, \ldots L+S$.

① ^1D $(L=2, S=0)$ → $J=2$

$^1D_2$ → $2\times2+1 = 5$ states.

$M_J = -2, -1, \ldots 2$

② ^3P $(L=1, S=1)$, ∴ $J = 0, 1, 2$

Notation $^3P_0$, $^3P_1$, $^3P_2$ states = 5 + 3 + 1 = 9.

③ ^1S ($L=0$, $S=0$) : $J = 0$

Notation : $^1S_0$.

Identifying the Ground Term (Hund's Rules)

(A) Maximum multiplicity:
The term with largest S, has the lowest energy

Here $S=1$ (^3P) is lower than $S=0$ (^1D, ^1S)

(B) Maximum L: If multiplicities are same, highest L wins.

(C) J-value: > If the shell is less than half-filled (p^2 is less than p^3), the state with the lowest J is the ground state.

> If shell is more than half-filled, the state with the highest J is the ground state,

since p-shell holds 6 e⁻, p^2 is less than half filled. Therefore we pick the lowest J from 3P multiplet

Ground term: $^3P_0$

(b) LS ground term of np^2 is $^3P_0$.

The Land'e interval rule gives,

$$\Delta E(J, J-1) = E(J) - E(J-1) = A_{LS} \cdot J$$

Show, $\dfrac{E(1) - E(0)}{E(2) - E(1)} = \dfrac{1}{2}$

$$\hat{H}_{LS} = A_{LS}\, \hat{L}\cdot\hat{S}$$

using $\hat{J}^2 = \hat{L}^2 + \hat{S}^2 + 2\hat{L}\cdot\hat{S}$

$$\langle \hat{L}\cdot\hat{S} \rangle = \frac{\hbar^2}{2}\left[J(J+1) - L(L+1) - S(S+1) \right]$$

$$\therefore E(J) = A_{LS}\frac{\hbar^2}{2}\left[J(J+1) - L(L+1) - S(S+1) \right]$$

This leads us to

$$\Delta E(J, J-1) = E(J) - E(J-1) = A_{LS} \cdot J$$

For ^3P, $E(1) - E(0) = A_{LS}$

$E(2) - E(1) = 2A_{LS}$

$\therefore \left| \frac{E(1)-E(0)}{E(2)-E(1)} \right| = \frac{1}{2}$

For carbon ($n=2$),

Determine A_{LS} for carbon,

predict $^3P_2 - {}^3P_1$ splitting.

Given $^3P_1 - {}^3P_0$ splitting $\sim 16.4\ cm^{-1}$

$E(1) - E(0) = A_{LS} = 16.4\ cm^{-1}$

Ans for carbon

$^3P_2 - {}^3P_1$ splitting

$= \Delta E(2,1) = E(2) - E(1)$

$= 2A_{LS}$

$= 2 \times 16.4 = 32.8\ cm^{-1}$

Is multiplet normal / inverted ?

Normal $\Rightarrow$ $A_{LS} > 0$ and $J = |L-S|$ has lowest energy

Inverted $\Rightarrow$ $A_{LS} < 0$ and $J = L+S$ has lowest energy

For Carbon ($2p^2$):

- Shell filling:
  - p-shell can hold 6 e⁻
  - carbon has 2, which is < half-filled
- ~~NOW~~ ~~For~~ shells. This means $\Rightarrow$ spin-orbit interaction follows a "normal" ordering where ground state has min possible J.

∴ For carbon this multiplet is NORMAL

lowest: $J=0 < (J=1) < (J=2)$

(e) Now consider Pb, $Z = 82$.

Configuration = $6p^2$.

In the jj coupling limit, each p-electron has definite $j = 1/2$ or $j = 3/2$.

Enumerate all the possible jj coupled states. $(j_1, j_2)\,J$ for two p-electrons, write spectroscopic labels.

Ans In heavy atoms (large Z), the spin-orbit coupling on each electron dominates over inter-electron repulsion.

Single Electron states: $j_i = l_i \pm 1/2 = 1/2$ for p-electron.
$= 3/2$

Two electrons combine as $(j_1, j_2)\,J$.

Possible (j_1, j_2) pairs for two p-electrons

(1) $(1/2, 1/2)$ – both e^- in lower spin-orbit level $p_{1/2}$.

(2) $(1/2, 3/2)$ – one in $p_{1/2}$, one in $p_{3/2}$.

(3) $(3/2, 3/2)$ – both in $p_{3/2}$.

Pauli Restriction in jj coupling

For equivalent electrons (same n, l, j), the analogue of Hund's restriction tells that **J must be "even"** the antisymmetrisation requirement for two identical j-fermions.

For non-equivalent electrons (different j), all J values from $|j_1 - j_2|$ to $j_1 + j_2$ are allowed.

Enumerating the allowed J states:

(1) $(1/2, 1/2)$ $\quad J \in \{0, 1\}$ $\quad \therefore$ **J = 0 only**

But J = even (As eq. electrons) State count: $2J + 1 = 1$

(iii) $(1/2, 3/2)$: Non equivalent electrons. (diff j)

$\Rightarrow$ All J are allowed.

$$J \in \{1, 2\}.$$

State count: $3 + 5 = 8$ states.

(iii) B $(3/2, 3/2)$: Equivalent electrons.

Naive coupling $J \in \{0, 1, 2, 3\}$.

Antisymmetry requires even J.

$$J \in \{0, 2\}$$

State count: $1 + 5 = 6$ states.

∴ Total state count = $1 + 8 + 16 = 15$

Total J-levels: $\{0, 0, 1, 2, 2\}$. — 5 levels.

Comparison with LS coupling levels.

Recall, LS coupling produced terms: ~~$^3P_{0,1,2}$, ^1D~~

$^1S_0$, $^3P_{0,1,2}$, $^1D_2$ $\Rightarrow$ i.e, J-levels ~~0,1,2~~ $\{0, 0, 1, 2, 2\}$

- $^1S_0$: $J = 0$
- $^3P_0$: $J = 0$
- $^3P_1$: $J = 1$
- ~~$^3P_1$: J = 1~~
- $^3P_2$: $J = 2$
- $^1D_2$: $J = 2$.

Both schemes yield exactly 5 J-levels with same J-distribution $\{0, 0, 1, 2, 2\}$.

Hilbert Space Verification

observe carefully, LS and jj coupling are two different bases, in the same 15-dimensional antisymmetrised, two-electron Hilbert space. The total $\hat{J}$ is conserved by the full atomic Hamiltonian (Coulomb + Spin-orbit + Relativistic Kinetic) —

But neither $\hat{L}, \hat{S}, \hat{j}_1$ nor $\hat{j}_2$ is exactly conserved.

The J eigenvalues (and their multiplicities) are basis-independent

→ only their labelling and the eigenvectors change.

In intermediate coupling the physical levels are linear combinations of LS- and jj-states sharing the same J.

Thus both schemes yield exactly i.e. 5 J-levels with the same J-distribution.

Verified

Spectroscopic Notation in jj coupling $(j_1, j_2)_J$

Configuration	J	Notation
$(p_{1/2})^2$	0	$(1/2, 1/2)_0$
$p_{1/2} \cdot p_{3/2}$	1	$(1/2, 3/2)_1$
	2	$(1/2, 3/2)_2$
$p_{1/2}, p_{3/2}$	0	$(3/2, 3/2)_0$
$(p_{3/2})^2$	2	$(3/2, 3/2)_2$

Energy ordering in pure jj coupling.

In the jj limit, energies are dominated by the spin-orbit splitting ζ_{np} (b/w $p_{1/2}$ and $p_{3/2}$)

Recall, $E(p_{3/2}) - E(p_{1/2}) = \frac{3}{2}\zeta_{np}$ for a single e⁻

So,

- $(p_{1/2})^2$ – lowest energy
- $p_{1/2} \cdot p_{3/2}$ – middle (one excited)
- $(p_{3/2})^2$ – highest (both in upper sublevel)

For lead, $(6p^2, Z=2)$. The spin-orbit splitting ζ_{6p} is enormous

5 (d) The spin-orbit parameter

$\zeta \propto Z^4$ > $K \propto Z$..

↓

electrostatic repulsion.

In intermediate coupling the levels are labelled by J only.

: Two levels of same J, arising from different LS terms.

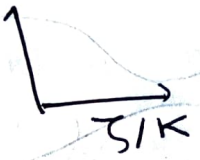
undergo an Avoided level crossing as

ζ / K increases.

Identify which pair of J-levels (from (a) - (c)) undergoes such a crossing in the np^2 sequence.

$K \rightarrow$ fixed.

Plot E



Am $\zeta/K << 1$ → pure LS coupling.

Levels ≡ $^{2S+1}L_J$

$\zeta/K >> 1$ → pure jj coupling levels ≡ $(j_1, j_2)_J$

However, in intermediate stage only J and parity remains good quantum numbers.

And levels are linear combination.

Now, which levels can mix? - Two levels can mix (i.e eigenvectors smoothly evolve and eng energies show avoided crossings), only if they share same exact symmetries that the full Hamiltonian conserves.

ie They must
- same J
- same parity
- same configuration. (np^2)

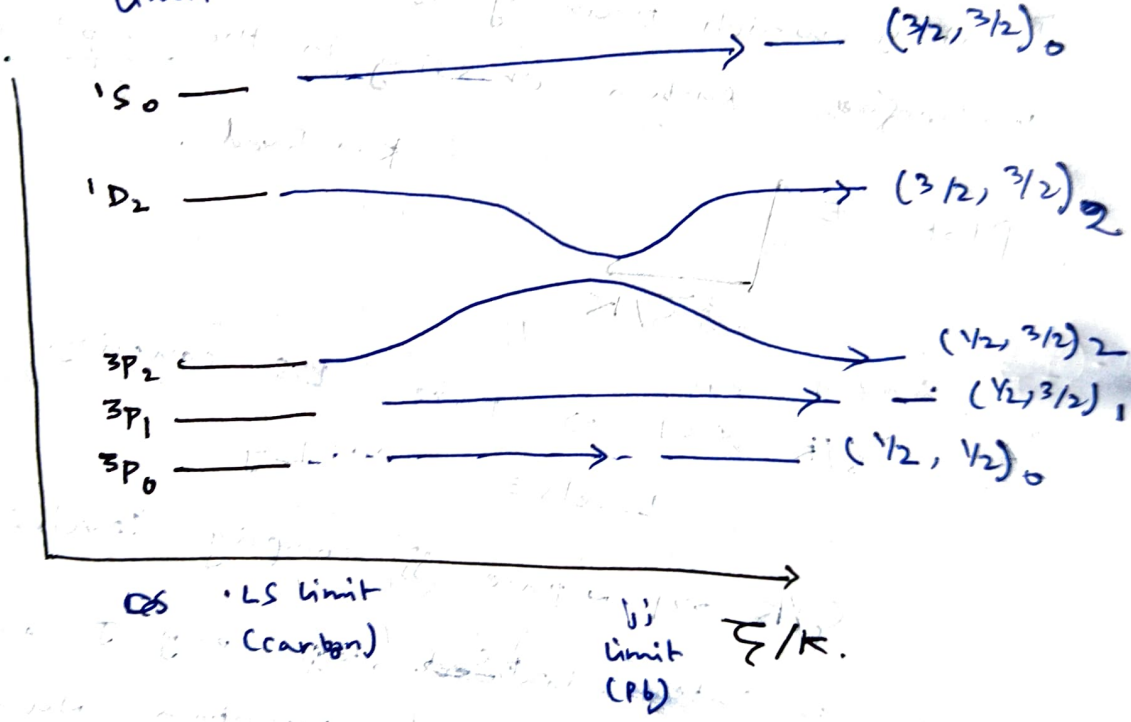
In np^2 configuration, (Among those earlier)

① $J = 0$ Pair : $^3P_0$ and $^1S_0$

② $J = 2$ Pair : $^3P_2$ and $^1D_2$

③ $J = 1$ Level : $^3P_1$ → This is stand alone, no level to cross with.

So, $(^3P_0, ^1S_0)$, $(^3P_2, ^1D_2)$ → both undergo level crossing as they evolve toward jj limit.



Preserved Quantum Numbers

The Quantum nos. corresponding to conserved symmetries remain constant

a) Total Angular Momentum – $\vec{J}$

b) Magnetic Quantum Number – M_J

c) Parity : The configuration np^2 always has parity $(-1)^{\Sigma l_i} = (-1)^2 = 1$ even.

Question 4 : Hyperfine Structure

(a) Write down

$$\hat{H}_{HF} = A\,\hat{I}\cdot\hat{J}$$

for $1s_{1/2}$ state of ~~energy~~ Hydrogen.

Find energies of $F=1$, $F=0$ state in terms of A.

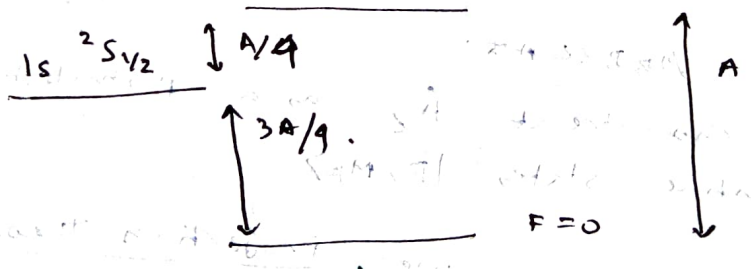
Ans Recall, in Hyperfine interaction $\vec{I}$, $\vec{J}$ can change direction. But, $\vec{F} = \vec{I} + \vec{J}$ → remains constant.

$$\therefore E_{HFS} = A\langle \vec{I}\cdot\vec{J}\rangle = \frac{A}{2}\{F(F+1) - I(I+1) - J(J+1)\}$$

Given Proton has spin $I = 1/2$ and $J = 1/2$ for $1s^2S_{1/2}$.

$\therefore$ $F=0$ and $F=1$ and these hyperfine levels have energies,

$$E_{HFS} = \frac{A}{2}\left\{F(F+1) - \frac{3}{4} - \frac{3}{4}\right\} = \begin{cases} A/4 & F=1 \\ -3A/4 & F=0 \end{cases}$$



$$\therefore \boxed{\Delta E_{HFS} = A} \quad (F=0 \rightarrow F=1)$$

$$\therefore A = h\nu$$

$$\therefore A = (1420 \times h)\ J \times 10^6$$

$$= \frac{1420 \times 6.62 \times 10^{-34} \times 10^6}{1.6 \times 10^{-19}}\ eV$$

or simply $A = 1420$ MHz.

$$= 5.875 \times 10^{-6}\ eV$$

center of gravity

$$\sum_F (2F+1)\, E_F = (2\times 0+1)\, E_0 + (2\times 1+1)\, E_1$$

$$= 1\times -\frac{3A}{4} + 3\times\left(\frac{A}{4}\right) = 0$$

proved

∴ Hyperfine interaction just redistribute energy levels without shifting the centroid — req. since trace $(\hat{I}\cdot\hat{J}) = 0$ when summed over full of $(2I+1)(2J+1)$ dim-space.

(b) In a weak magnetic field ($\mu_B B \ll A\hbar$), F is a good quantum number.

Derive the energy of each $|F, M_F\rangle$ sublevel to first order in B.

The Hamiltonian reads $\hat{H}_Z = -\hat{\mu}\cdot\vec{B}$

$$= \frac{\mu_B}{\hbar}\,(g_J\,\hat{J} + g_I\,\hat{I})\cdot\vec{B}$$

Since, $\mu_B B \ll A\hbar$.

we can treat $\hat{H}_Z$ as a perturbation on hyperfine states $|F, M_F\rangle$.

To first order, we use projection theorem, which states that the total magnetic moment is proportional to the total angular momentum $\vec{F}$.

∴ Energy of sublevels

Let $\vec{B} = B\hat{z}$

Calc. 1st order energy shift.

$$E_{z0}^{(1)} = \langle F, M_F|\, \hat{H}_z\, |F, M_F\rangle$$

$$= \frac{\mu_B}{\hbar} B\, \langle F, M_F|\,(g_J\,\hat{J} + g_I\,\hat{I})\,|F, M_F\rangle$$

$$= \frac{\mu_B\, g_F\, B}{\hbar}\, \langle F, M_F|\,\hat{F}_z\,|F, M_F\rangle = g_F \mu_B B M_F$$

Add this to the zero-field energies $E_F^{(0)}$ derived.

State	F	M_F	$E(F, M_F)$	$\|F, M_F\rangle$
Triplet	1	+1	$A/4 + g_F \mu_B B$	$\|1, 1\rangle$
		0	$A/4$	$\|1, 0\rangle$
		−1	$A/4 - g_F \mu_B B$	$\|1, -1\rangle$
singlet	0	0	$-3A/4$	$\|0, 0\rangle$

Derive expression for g_F

The total magnetic moment operator is

$$\hat{\mu} = -\frac{g_J \mu_B \hat{J}}{\hbar} - \frac{g_I \mu_N \hat{I}}{\hbar}$$

($\hat{J}$: e^-; $\hat{I}$: Nuclear)

since $\mu_N = \mu_B \left(\frac{m_e}{m_p}\right) \simeq \mu_B / 1836 \ll \mu_B$.

$$\therefore \hat{\mu} \simeq -\frac{g_J \mu_B \hat{J}}{\hbar} \quad \text{(satisfy)}.$$

($\hat{F}$ → cont; $\hat{J}$ eff. ≐ proj. on $|M_F, F\rangle$ ie)

Acc. to Wigner-Eckart projection:

For a vector operator $\hat{V}$, within a state of definite F,

$$\langle F, M_F | \hat{V} | F, M_F \rangle = \frac{\langle \hat{V} \cdot \hat{F} \rangle}{\hbar^2 F(F+1)} \langle F, M_F | \hat{F} | F, M_F \rangle$$

when $\hat{V} = \hat{J}$

$$\langle J \cdot \hat{F} \rangle = \frac{1}{2} [F(F+1) + J(J+1) - I(I+1)] \hbar^2$$

∴ Zeeman energy is

$$E_z = g_F \mu_B B M_F \quad \text{with}$$

$$g_F = g_J \frac{F(F+1) + J(J+1) - I(I+1)}{2F(F+1)}$$

We have defined g_F as

$$\hat{H}_z \equiv g_F \frac{\mu_B}{\hbar} \hat{F}\cdot\vec{B} = g_J \frac{\mu_B}{\hbar} \hat{J}\cdot\vec{B}$$

($\hat{F}\cdot\vec{B}$: constant; $\hat{J}\cdot\vec{B}$: precessing)

Evaluation

(i) $(F=1,\ J=1/2,\ I=1/2)$

$$g_{F=1} = 2 \times \frac{1(2) + \frac{1}{2}\left(\frac{3}{2}\right) - \frac{1}{2}\left(\frac{3}{2}\right)}{2(1)(2)}$$

$$= 2 \times \frac{2}{4} = 1$$

Result $F=1$ levels split with g-factor $=1$.

$$\therefore \Delta E = \mu_B B M_F$$

(ii) $F=0$

$\rightarrow$ $g_{F=0} \rightarrow$ DNE.

$\therefore$ $F=0 \rightarrow$ zero Total Angular Momentum,

The expectation value of any vector operation operator ($\hat{F}$ on $\hat{\mu}$) = zero in $F=0$ state.

$\therefore$

F	$\Delta E_F^{(1)}$ shift.
1	$\mu_B B$
	0
	$-\mu_B B$
0	0

4 (c) Paschen Back limit for hyperfine structure — $\mu_B B \gg A\hbar$.

Good quantum no.s are m_J and m_I.

Write energy eigenvalues in this regime, and enumerate all 4 states using (m_J, m_I).

Ans ⇒ When $\mu_B B \gg A\hbar$

or $B \gg \frac{A\hbar}{\mu_B} \simeq \frac{1420 \times 10^6 \times 6.63 \times 10^{-34}}{2\pi \times 9.27 \times 10^{-24}} \sim 10^{-1} T$

The Zeeman term dominates and decouples I from J.

Good quantum no.s become m_I, m_J

∴ The hamiltonian becomes diagonal in $|m_I, m_J\rangle$.

with energies

$$E(m_J, m_I) = g_J \mu_B B m_J + A m_J m_I$$

1st order correction

(where Nuclear Zeeman term $g_I \mu_N B m_I$ is smaller, and it is neglected)

4 states $m_J = \pm 1/2$, $m_I = \pm 1/2$ [$J = 1/2$, $I = 1/2$]

m_J	m_I	Energy	Type of Field seeking
$1/2$	$1/2$	$\frac{1}{2}\mu_B B + \frac{A}{4}$	weak
$1/2$	$-1/2$	$\frac{1}{2}\mu_B B - \frac{A}{4}$	weak
$-1/2$	$1/2$	$-\frac{1}{2}\mu_B B - \frac{A}{4}$	strong
$-1/2$	$-1/2$	$-\frac{1}{2}\mu_B B + \frac{A}{4}$	strong

Weak field seeking states (Energy ↑ B↑) are repelled from high-field regions and can be confined in magnetic minimum field traps — achievement of BEC of H.

But strong field seeking states (E ↓ B ↑), since Earnshaw's Theorem prevents a local maximum of magnetic field in free space, these atoms are pulled towards a magnets and cannot be trapped in static fie

(3) For states connecting to $|F=1, M_F=0\rangle$ and $|F=0, M_F=0\rangle$ at zero field, show that their exact energies at an arbitrary field B can be written as

$$E = -\frac{A}{4} \pm \frac{A}{2}\sqrt{1+x^2} \qquad x = \frac{2\mu_B B}{A\hbar}.$$

Breit Rabi Formula for $(M_F = 0)$

We consider the basis of uncoupled states $|m_J, m_I\rangle$.

For $M_F = 0$, there are 2 states.

$$|\phi_1\rangle = |1/2, -1/2\rangle$$

$$|\phi_2\rangle = |-1/2, 1/2\rangle.$$

Hamiltonian is, $\hat{H} = A\,\hat{I}\cdot\hat{J} + (g_J \mu_B \hat{J}_z + g_I \mu_B \hat{I}_z)B.$

Neglecting, very small nuclear term ($g_I \approx 0$), $g_J \approx 2$.

Matrix in $\{|\phi_1\rangle, |\phi_2\rangle\}$ basis is.

$$H = \begin{bmatrix} -\frac{A}{4} + \mu_B B & \frac{A}{2} \\ \frac{A}{2} & -\frac{A}{4} - \mu_B B \end{bmatrix}$$

The eigenvalues E are found by solving $\det(H - EI) = 0$

$$\left(-\frac{A}{4} - E + \mu_B B\right)\left(-\frac{A}{4} - E - \mu_B B\right) = \left(\frac{A}{2}\right)^2$$

$$\Rightarrow \quad \left(E + \frac{A}{4}\right)^2 - (\mu_B B)^2 = \frac{A^2}{4}$$

$$\left(E + \frac{A}{4}\right)^2 = \frac{A^2}{4} + (\mu_B B)^2 = \frac{A^2}{4}\left[1 + \left(\frac{2\mu_B B}{A}\right)^2\right].$$

$$\Rightarrow \quad \boxed{E = -\frac{A}{4} \pm \frac{A}{2}\sqrt{1+x^2}}$$

with $x = \dfrac{2\mu_B B}{A}$.

Limits

- $B = 0$: $E_+ = -\frac{A}{4} + \frac{A}{2} = +\frac{A}{4}$ (the $F=1$ level);

$E_- = -\frac{A}{4} - \frac{A}{2} = -\frac{3A}{4}$ (the $F=0$ level).

- $B \to \infty$: $E_\pm \to -\frac{A}{4} \pm \frac{1}{2} \mu_B B$.

The states adiabatically connect to the high-field $|m_J = \pm 1/2, m_I = \mp 1/2\rangle$ states.

At what field strength does the frequency of the transition have its minimum.

$|F=1, M_F = 0\rangle \leftrightarrow |F=0, M_F = 0\rangle$

sensitivity to B?

First, we calculate the transition frequency b/w the two $M_F = 0$ level.

$$h\nu = E_+ - E_- = A\sqrt{1+x^2}$$

$$\to \nu = \frac{A}{h}\sqrt{1+x^2}$$

Minimum sensitivity to B

First order sensitivity is.

$$\frac{\partial \nu}{\partial B} = \frac{A}{h} \frac{x}{\sqrt{1+x^2}} \cdot \frac{2\mu_B}{A\hbar}$$

clearly $\frac{\partial \nu}{\partial B} = 0 \Rightarrow x = 0 \Rightarrow B = 0$.

So, the $|F=1, M_F=0\rangle \leftrightarrow |F=0, M_F=0\rangle$ transition is field insensitive (upto first order) at $B = 0$.

Explain why this is the operating point for hydrogen masers and atomic clocks.

(i) $M_F = 0$ transition has no first order shift with B. (In contrast with $M_F = \pm 1$ states)

(ii) Because shift is only quadratic (B^2), it is not affect due to tiny magnetic field fluctuations in the lab — like stray fields from currents, magnetised metals, the Earth's field. That would otherwise broaden and shift the clock frequency