On the Feynman path-integral approach to the Aharonov-Bohm effect

Submitted by

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Contents

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ABSTRACT

In this report, we present a comprehensive discussion on the famous *Aharonov-Bomb* Effect (Aharonov & Bohm (1959)), by setting up Feynman's path integral formalism and its applying it to solve the AB effect. In particular, we examine the detailed calculations provided by Gerry & Singh (1979) for the propagator ($K(\mathbf{r}'', \mathbf{r}'; \tau)$)for an electron traveling through a region where the magnetic field is zero, but the vector potential is non-zero. Our analysis demonstrates that both traditional methods and those based on path integrals can successfully account for the observed shift in the interference pattern when the magnetic flux (Φ_B) of the solenoid.

Chapter 1

Literature Review

Aharonov and Bohm in their 1959 paper (Aharonov & Bohm (1959)) introduced the effect that now bears their name, proposing a double-slit interference experiment in which electrons acquire a phase shift due to a magnetic flux confined to an inaccessible solenoid. Despite traversing regions of zero magnetic field, electrons experience a relative phase shift proportional to the enclosed flux. This phase shift was later confirmed experimentally and is widely regarded as a quintessential manifestation of quantum nonlocality and the physical reality of gauge potentials.

Topological Decomposition in Path Integrals The Feynman path-integral formulation naturally incorporates the global features of the AB effect. When a particle propagates in a multiply-connected space, such as the punctured plane, its path integral decomposes into a sum over homotopy classes. Schulman (1971) and Laidlaw & DeWitt (1971) demonstrated that the quantum propagator in such spaces is a weighted sum over inequivalent homotopy classes, with weights determined by one-dimensional unitary representations of the fundamental group.

In the AB setup, the fundamental group of the configuration space is $\pi_1 \cong \mathbb{Z}$, corresponding to integer winding numbers around the solenoid. The propagator takes the form:

$$K = \sum_{n=-\infty}^{\infty} \chi_n K_n, \tag{1.1}$$

where K_n is the amplitude for paths winding n times around the solenoid, and $\chi_n = \exp(i\alpha n)$ is the phase factor associated with the flux Φ via $\alpha = q\Phi/(\hbar c)$. This decomposition reveals the topological nature of the AB phase, dependent only on the winding number.

Explicit Path-Integral Calculations Gerry & Singh (1979) provided a detailed path-integral calculation of the AB propagator using polar coordinates and a periodic constraint on the angular coordinate. This constraint ensures that paths are correctly classified by their winding numbers, allowing the separation of the propagator into contributions from each topological sector:

$$K(r'', \theta''; r', \theta'; \tau) = \sum_{n=-\infty}^{\infty} e^{i\alpha(\theta'' - \theta' + 2\pi n)} K_n(r'', r'; \tau). \tag{1.2}$$

Each K_n resembles the free-particle propagator in two dimensions, modified by a phase from the vector potential. Without the periodic constraint, all homotopy classes would yield identical results, and the AB effect would vanish.

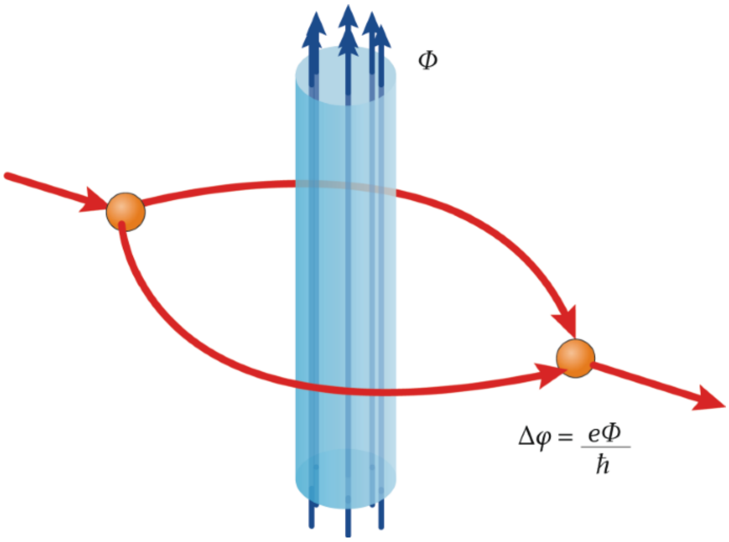
Inomata & Singh (1978b)took a similar approach using angular constraints and Fourier expansions. Their treatment also produced a decomposition into winding sectors and showed that the propagator depends on modified Bessel functions $I_{|n+\alpha|}$.

Gauge Invariance and Quantum Nonlocality Early work by Edwards (1967) emphasized that the quantum propagator remains gauge invariant: only the holonomy $\oint A \cdot dr$ around non-contractible loops affects physical predictions. The AB effect, in this view, arises from the nontrivial topology of the configuration space and not from any local action of the field. The phase acquired by the electron is a Wilson loop, a gauge-invariant object.

To conclude, the literature on the AB effect via path integrals consistently reveals that the observable phase shift is a topological phenomenon tied to the winding of particle trajectories. The decomposition into homotopy classes, weighted by one-dimensional unitary representations of the fundamental group, elegantly captures the global influence of electromagnetic potentials. The path-integral framework not only reproduces known results but also clarifies the deep topological structure underlying quantum gauge phenomena. Mouchet (2022) summarizes this insight as a "major breakthrough," noting that the Feynman integral "opened a direct window on quantum properties of topological origin in the configuration space"

Chapter 2

Introduction



The Aharonov–Bohm (AB)[1] effect demonstrates that electromagnetic potentials can exert physical influence in quantum mechanics, even in regions where the associated electric and magnetic fields vanish. In the classic setup, electrons travel along two distinct paths that encircle a solenoid containing a confined magnetic field. Although the electrons never enter the region where the magnetic field is present, the interference pattern observed on a detection screen shifts depending on the magnetic flux enclosed by the paths. This phase shift cannot be explained by classical electromagnetic theory, in which only fields are believed to produce observable effects. The AB effect thus reveals the fundamental role of gauge potentials in quantum mechanics, showing that the vector potential $\mathbf{A}$ can yield measurable consequences even in field-free regions.

The path integral formulation of quantum mechanics, introduced by Feynman, offers a natural framework for analyzing the AB effect. In this approach, the probability amplitude for a particle to travel from one point to another is given by a sum over all possible paths, each weighted by a phase factor $\exp(iS/\hbar)$, where S is the classical action. When a vector potential $\mathbf{A}$ is present, the action includes a term $q\int \mathbf{A}\cdot \mathrm{d}\mathbf{x}$, which introduces a path-dependent phase. In the AB setup, although the magnetic field vanishes along each path, the line integral of the vector potential around the loop enclosing the solenoid leads to a nonzero phase difference between paths. This difference is directly proportional to the enclosed magnetic flux, providing a natural explanation for the observed interference shift. Thus, the path integral method elegantly captures the global and topological features of the AB effect through the accumulated phase over classes of paths.

Homotopy theory provides deeper insight into the topological nature of the AB effect. The key point is that the space through which the electrons propagate is not simply connected—the interior of the

[1] Video demonstration: Wolfram Alpha. Here upon increasing the electron's momentum (for a fixed $\mathbf{B}_{sol}$) the fringe width (β) shrinks, as $p_{e^-} \propto 1/\lambda_D \propto 1/\beta$

solenoid is excluded from the configuration space. As a result, paths that encircle the solenoid cannot be continuously deformed into one another without crossing the excluded region. Homotopy theory classifies such inequivalent paths into distinct equivalence classes based on their winding number around the solenoid. The interference pattern observed in the AB effect depends on these winding numbers, since different homotopy classes correspond to different accumulated quantum phases. This topological characterization explains why the effect depends solely on the total magnetic flux enclosed and not on the detailed field configuration, highlighting the significance of global spatial features in quantum behavior.

Feature	Why Path Integral Formalism Helps
Phase from vector potential A	The path integral includes the term $\int \dot{x} \cdot \mathbf{A}\,\mathrm{d}t$ in the action, so even if the magnetic field $\mathbf{B} = 0$, the vector potential affects the phase.
No force, still observable effect	Classical force depends on $\mathbf{B}$, but the quantum phase depends on $\mathbf{A}$, showing that the electron is affected even in force-free regions.
Interference between different paths	The path integral naturally sums over all possible trajectories; each path contributes a phase $\exp(iS[x]/\hbar)$, leading to observable interference patterns.
Topological effects (e.g., winding)	Paths that wind around the solenoid different numbers of times acquire different phases. The path integral captures this through topologically distinct classes of paths.

Table 2.1: Importance of the Path Integral Formalism in the Aharonov-Bohm Effect

Chapter 3

The Feynman Path Integral Formulation

3.1 Physical Intuition

Take the two-slit experiment. Each time an electron hits the screen, there is no way to tell which slit the electron has gone through. After repeating the same experiment many times, a fringe pattern gradually appears on the screen, proving that there is interference between two waves—one from each slit. We conclude that we need to sum amplitudes of these two waves that correspond to different paths of the electron. Now imagine making more slits. There are more paths, each contributing an amplitude. As the number of slits increases, the obstruction effectively disappears. Yet many paths still contribute to the amplitude for the electron to reach the screen.

As we generalize this thought experiment, we are led to conclude that the amplitude for a particle to move from x_i to x_f includes contributions from many paths. One such path is the classical trajectory, but many nonclassical paths contribute as well. This leads to the notion of the path integral:

$$\langle x_f, t_f | x_i, t_i \rangle = \int \mathcal{D}x(t)\, e^{iS[x(t)]/\hbar}, \tag{3.1}$$

where the weight $e^{iS[x(t)]/\hbar}$ is the classical action:

$$S[x(t)] = \int_{t_i}^{t_f} \mathrm{d}t\, L(x(t), \dot{x}(t)). \tag{3.2}$$

In the $\hbar \to 0$ limit, $e^{iS/\hbar}$ oscillates rapidly, canceling contributions from most paths. Only paths near those making the action stationary remain. These are classical trajectories. When $\hbar$ is small but finite, the particle explores nearby paths whose action differs by roughly $\hbar$.

3.2 Hamilton-Jacobi Equation from Path Integral Formulation

Let us begin by expressing the wavefunction ψ in polar form as:

$$\psi(x,t) = \sqrt{\rho(x,t)}\, \exp\left(\frac{i}{\hbar} S(x,t)\right), \tag{3.3}$$

where $\rho(x,t) = \psi^*(x,t)\psi(x,t)$ is the probability density, and $S(x,t)$ is the classical action, playing the role of a phase.

Now substitute this form of ψ into the time-dependent Schrödinger equation:

$$i\hbar \frac{\partial \psi}{\partial t} = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right]\psi. \tag{3.4}$$

We compute the time derivative:

$$\frac{\partial \psi}{\partial t} = \left(\frac{1}{2\sqrt{\rho}}\frac{\partial \rho}{\partial t} + \frac{i}{\hbar}\frac{\partial S}{\partial t}\sqrt{\rho}\right)\exp\left(\frac{i}{\hbar}S\right). \tag{3.5}$$

Compute the second spatial derivative:

$$\frac{\partial^2\psi}{\partial x^2} = \left[\frac{1}{2\sqrt{\rho}}\frac{\partial^2\rho}{\partial x^2} - \frac{1}{4\rho^{3/2}}\left(\frac{\partial\rho}{\partial x}\right)^2 + \frac{i}{\hbar}\frac{\partial^2 S}{\partial x^2}\sqrt{\rho} + \frac{2i}{\hbar}\frac{\partial S}{\partial x}\frac{\partial\sqrt{\rho}}{\partial x} - \frac{1}{\hbar^2}\left(\frac{\partial S}{\partial x}\right)^2\sqrt{\rho}\right]\exp\left(\frac{i}{\hbar}S\right). \tag{3.6}$$

Substitute into the Schrödinger equation and collect real and imaginary parts. The real part yields:

$$-\frac{\partial S}{\partial t} = \frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 + V(x) - \frac{\hbar^2}{2m}\frac{1}{\sqrt{\rho}}\frac{\partial^2\sqrt{\rho}}{\partial x^2}. \tag{3.7}$$

Now take the classical limit $\hbar \to 0$, so the last term (quantum potential) vanishes. We obtain:

$$-\frac{\partial S}{\partial t} = \frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 + V(x), \tag{3.8}$$

which is precisely the classical Hamilton-Jacobi equation:

$$\frac{\partial S}{\partial t} + H\left(x, \frac{\partial S}{\partial x}\right) = 0, \tag{3.9}$$

with the Hamiltonian $H(x,p) = \frac{p^2}{2m} + V(x)$ and $p = \partial S/\partial x$.

This shows that in the limit $\hbar \to 0$, the Schrödinger equation reduces to the classical Hamilton-Jacobi equation.

Path Integral as a Lagrangian Formulation: A derivation for the the kernel

It starts with the amplitude for a particle to go from x_i at t_i to x_f at t_f, and sums over all intermediate configurations:

We begin with the expression for the short-time propagator in the momentum representation:

$$\begin{aligned}\langle x_1, t+\Delta t | x_0, t\rangle &= \int \frac{dp}{2\pi\hbar}\, e^{ipx_1/\hbar}\, e^{-i(px_0 + \frac{p^2}{2m}\Delta t + V(x_0)\Delta t + O(\Delta t^2))/\hbar} \\ &= \int \frac{dp}{2\pi\hbar}\, \exp\left\{\frac{i}{\hbar}\left[p(x_1 - x_0) - \frac{p^2}{2m}\Delta t - V(x_0)\Delta t + O(\Delta t^2)\right]\right\}.\end{aligned}$$

We separate the p-independent terms from the integral:

$$= e^{-\frac{i}{\hbar}(V(x_0)\Delta t + O(\Delta t^2))}\int \frac{dp}{2\pi\hbar}\exp\left[\frac{i}{\hbar}\left(p(x_1 - x_0) - \frac{p^2}{2m}\Delta t\right)\right].$$

Now, complete the square in the exponent:

$$\begin{aligned}p(x_1 - x_0) - \frac{p^2}{2m}\Delta t &= -\frac{\Delta t}{2m}\left(p^2 - \frac{2m}{\Delta t}(x_1 - x_0)p\right) \\ &= -\frac{\Delta t}{2m}\left(p - \frac{m(x_1 - x_0)}{\Delta t}\right)^2 + \frac{m(x_1 - x_0)^2}{2\Delta t}.\end{aligned}$$

So the exponent becomes:

$$\frac{i}{\hbar}\left[-\frac{\Delta t}{2m}\left(p - \frac{m(x_1 - x_0)}{\Delta t}\right)^2 + \frac{m(x_1 - x_0)^2}{2\Delta t}\right].$$

Change variables: $p' = p - \frac{m(x_1 - x_0)}{\Delta t}$. The integral becomes:

$$\int \frac{dp}{2\pi\hbar}\exp\left[-\frac{i\Delta t}{2m\hbar}\left(p - \frac{m(x_1 - x_0)}{\Delta t}\right)^2\right] = \int \frac{dp'}{2\pi\hbar}\exp\left[-\frac{i\Delta t}{2m\hbar}p'^2\right].$$

This is a Fresnel integral:

$$\int_{-\infty}^{\infty} \exp\left[-\frac{i\Delta t}{2m\hbar}p'^2\right] dp' = \sqrt{\frac{2\pi m\hbar}{i\Delta t}}.$$

So the full expression becomes:

$$\langle x_1, t+\Delta t|x_0, t\rangle = \sqrt{\frac{m}{2\pi i\hbar\Delta t}} \exp\left[\frac{i}{\hbar}\left(\frac{m(x_1-x_0)^2}{2\Delta t} - V(x_0)\Delta t + O(\Delta t^2)\right)\right].$$

$$\langle x_f, t_f|x_i, t_i\rangle = \lim_{N\to\infty}\int \prod_{j=1}^{N-1} \mathrm{d}x_j \prod_{k=0}^{N-1} \langle x_{k+1}, t_{k+1}|x_k, t_k.\rangle \tag{3.10}$$

Using the short-time kernel derived from the Lagrangian:

$$\langle x_{k+1}, t_{k+1}|x_k, t_k\rangle = \left(\frac{m}{2\pi i\hbar\Delta t}\right)^{1/2} \exp\left[\frac{i}{\hbar}L(x_k, \dot{x}_k)\Delta t\right], \tag{3.11}$$

This is the Lagrangian formulation of quantum mechanics via path integrals, encoding dynamics through actions associated with each trajectory.

Now divide time into N intervals $\Delta t = (t_f - t_i)/N$, and insert $N-1$ completeness relations:

$$\langle x_f, t_f|x_i, t_i\rangle = \int \prod_{j=1}^{N-1} dx_j \prod_{k=0}^{N-1} \langle x_{k+1}, t_{k+1}|x_k, t_k\rangle \tag{3.12}$$

$$= \left(\frac{m}{2\pi i\hbar\Delta t}\right)^{N/2} \int \prod_{j=1}^{N-1} \mathrm{d}x_j \exp\left[\frac{i}{\hbar}\sum_{k=0}^{N-1}\left(\frac{m}{2}\left(\frac{x_{k+1}-x_k}{\Delta t}\right)^2 - V(x_k)\right)\Delta t\right] \tag{3.13}$$

Taking the limit $N \to \infty$, $\Delta t \to 0$, we obtain:

$$\boxed{K(x_f, t_f; x_i, t_i) \equiv \langle x_f, t_f|x_i, t_i\rangle = \int \mathcal{D}[x(t)]\, e^{\frac{i}{\hbar}S[x(t)]}} \tag{3.14}$$

3.3 The Propagator

The quantity

$$K(x_f, t_f; x_i, t_i) = \langle x_f, t_f|x_i, t_i\rangle, \tag{3.15}$$

is called the propagator. It determines how a wavefunction propagates:

$$\psi(x_f, t_f) = \int \mathrm{d}x_i\, K(x_f, t_f; x_i, t_i)\psi(x_i, t_i). \tag{3.16}$$

In other words it is the Green's function for the time dependent schrodinger equation.

If the Hamiltonian is time-independent, the propagator can be written as:

$$K(x_f, t_f; x_i, t_i) = \langle x_f|e^{-iH(t_f-t_i)/\hbar}|x_i\rangle \tag{3.17}$$

$$= \sum_n \langle x_f|n\rangle e^{-iE_n(t_f-t_i)/\hbar}\langle n|x_i\rangle \tag{3.18}$$

$$= \sum_n e^{-iE_n(t_f-t_i)/\hbar}\psi_n^*(x_f)\psi_n(x_i) \tag{3.19}$$

This allows us to extract the energy spectrum via Fourier analysis.

3.4 Euler lagrange equation from Path integrals

Under a small variation $\delta x(t)$ with fixed endpoints we have $(\delta x(t_i) = \delta x(t_f) = 0)$:

$$\int \mathcal{D}x(t)\, e^{\frac{i}{\hbar}S[x(t)+\delta x(t)]} = \int \mathcal{D}x(t)\, e^{\frac{i}{\hbar}S[x(t)]}. \tag{3.20}$$

Subtracting both sides:

$$\int \mathcal{D}x(t) \left(e^{\frac{i}{\hbar}S[x(t)+\delta x(t)]} - e^{\frac{i}{\hbar}S[x(t)]}\right) = 0. \tag{3.21}$$

Expand the exponential to first order:

$$e^{\frac{i}{\hbar}S[x(t)+\delta x(t)]} \approx e^{\frac{i}{\hbar}S[x(t)]} \left(1 + \frac{i}{\hbar}\delta S\right). \tag{3.22}$$

Substitute into (1):

$$\int \mathcal{D}x(t)\, e^{\frac{i}{\hbar}S[x(t)]} \frac{i}{\hbar}\delta S = 0. \tag{3.23}$$

The variation of the action is(for simplicity we assume this is in 1 dimension):

$$\delta S = \int_{t_i}^{t_f} \mathrm{d}t \left(\frac{\partial L}{\partial x}\delta x + \frac{\partial L}{\partial \dot{x}}\delta \dot{x}\right). \tag{3.24}$$

Integrate $\delta \dot{x}$ by parts (boundary terms vanish):

$$\delta S = \int_{t_i}^{t_f} \mathrm{d}t \left(\frac{\partial L}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}}\right) \delta x(t). \tag{3.25}$$

Now by substitution,

$$\frac{i}{\hbar} \int \mathcal{D}x(t)\, e^{\frac{i}{\hbar}S[x(t)]} \int_{t_i}^{t_f} \mathrm{d}t \left(\frac{\partial L}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}}\right) \delta x(t) = 0. \tag{3.26}$$

Since $\delta x(t)$ is arbitrary:

$$\left\langle \frac{\partial L}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}} \right\rangle = 0. \tag{3.27}$$

Equation (6) implies the Euler-Lagrange equation holds as an expectation value:

$$\left\langle \frac{\partial L}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}} \right\rangle = 0. \tag{3.28}$$

This matches Ehrenfest's theorem, linking quantum expectation values to classical dynamics.

3.5 Schrödinger Equation from Path Integral

We shift the trajectory $x(t)$ by a small amount $x(t) + \delta x(t)$ with the boundary condition that x_i is held fixed $(\delta x(t_i) = 0)$ while x_f is varied $(\delta x(t_f) \neq 0)$. Under this variation, the propagator changes by

$$\langle x_f + \delta x(t_f), t_f | x_i, t_i \rangle - \langle x_f, t_f | x_i, t_i \rangle = \frac{\partial}{\partial x_f} \langle x_f, t_f | x_i, t_i \rangle \delta x(t_f). \tag{3.29}$$

On the other hand, the path integral changes by

$$\int \mathcal{D}x(t)\, e^{iS[x(t)+\delta x(t)]/\hbar} - \int \mathcal{D}x(t)\, e^{iS[x(t)]/\hbar} = \int \mathcal{D}x(t)\, e^{iS[x(t)]/\hbar} \frac{i\delta S}{\hbar}. \tag{3.30}$$

Recall that the action changes by

$$\delta S = S[x(t)+\delta x(t)] - S[x(t)] = \int_{t_i}^{t_f} \mathrm{d}t \left(\frac{\partial L}{\partial x}\delta x + \frac{\partial L}{\partial \dot{x}}\delta \dot{x}\right) = \frac{\partial L}{\partial \dot{x}}\delta x\Big|_{t_i}^{t_f} + \int_{t_i}^{t_f} \mathrm{d}t \left(\frac{\partial L}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}}\right) \delta x. \tag{3.31}$$

The last terms vanishes because of the equation of motion (which holds as an expectation value, as we saw in the previous section), and we are left with

$$\delta S = \frac{\partial L}{\partial \dot{x}} \delta x(t_f) = p(t_f)\delta x(t_f) \tag{3.32}$$

By putting (3.31) and (3.28) together, and dropping $\delta x(t_f)$, we find

$$\frac{\partial}{\partial x_f} \langle x_f, t_f | x_i, t_i \rangle = \int \mathcal{D}x(t)\, e^{iS[x(t)]/\hbar} \frac{i}{\hbar} p(t_f). \tag{3.33}$$

This is precisely how the momentum operator is represented in the position space.

Now the Schrödinger equation can be derived by taking a variation with respect to t_f.

$$\frac{\partial S}{\partial t_f} = -H(t_f) \tag{3.34}$$

after using the equation of motion. Therefore,

$$\frac{\partial}{\partial t_f} \langle x_f, t_f | x_i, t_i \rangle = \int \mathcal{D}x(t)\, e^{iS[x(t)]/\hbar} \frac{-i}{\hbar} H(t_f). \tag{3.35}$$

If

$$H = \frac{p^2}{2m} + V(x), \tag{3.36}$$

the momentum can be rewritten using Eq. (3.31), and we recover the Schrödinger equation,

$$i\hbar \frac{\partial}{\partial t} \langle x_f, t_f | x_i, t_i \rangle = \left(\frac{1}{2m} \left(\frac{\hbar}{i} \frac{\partial}{\partial x_f} \right)^2 + V(x_f) \right) \langle x_f, t_f | x_i, t_i \rangle. \tag{3.37}$$

In other words, the path integral contains the same information as the conventional formulation of the quantum mechanics.

Chapter 4

Setting up

4.1 Lagrangian of an Electron in an Electromagnetic Field

The Lagrangian for a free particle of mass m is given by:

$$\mathcal{L}_0 = \frac{1}{2} m\dot{\mathbf{r}}^2, \tag{4.1}$$

where $\dot{\mathbf{r}}$ is the velocity.

To describe a charged particle (e.g., an electron with charge $q = -e$) in the presence of electromagnetic fields, we include the effects of the scalar potential $\phi(\mathbf{r}, t)$ and vector potential $\mathbf{A}(\mathbf{r}, t)$.

The interaction of the particle with the electromagnetic field is captured by the minimal coupling prescription, and the Lagrangian becomes:

$$\boxed{\mathcal{L} = \frac{1}{2} m\dot{\mathbf{r}}^2 + q\dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}, t) - q\phi(\mathbf{r}, t)} \tag{4.2}$$

This Lagrangian leads to the correct equations of motion via the Euler–Lagrange equation:

$$\frac{d}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{r}}}\right) - \frac{\partial \mathcal{L}}{\partial \mathbf{r}} = 0. \tag{4.3}$$

We compute:

$$\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{r}}} = m\dot{\mathbf{r}} + q\mathbf{A}, \tag{4.4}$$

$$\frac{d}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{r}}}\right) = m\ddot{\mathbf{r}} + q\left(\frac{\partial \mathbf{A}}{\partial t} + (\dot{\mathbf{r}} \cdot \nabla)\mathbf{A}\right), \tag{4.5}$$

$$\frac{\partial \mathcal{L}}{\partial \mathbf{r}} = q\dot{\mathbf{r}} \cdot \nabla\mathbf{A} - q\nabla\phi. \tag{4.6}$$

Using the identities:

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}, \tag{4.7}$$

$$\mathbf{B} = \nabla \times \mathbf{A}, \tag{4.8}$$

we derive the Lorentz force law:

$$m\ddot{\mathbf{r}} = q\left(\mathbf{E} + \dot{\mathbf{r}} \times \mathbf{B}\right). \tag{4.9}$$

Thus, the Lagrangian

$$\mathcal{L} = \frac{1}{2} m\dot{\mathbf{r}}^2 + q\dot{\mathbf{r}} \cdot \mathbf{A} - q\phi \tag{4.10}$$

fully describes the dynamics of a charged particle in an external electromagnetic field and leads to the correct classical equations of motion.

In classical mechanics, the action is given by

$$S = \int \mathcal{L}(q, \dot{q}, t)\, \mathrm{d}t. \tag{4.11}$$

If we modify the Lagrangian by adding a total time derivative of a function $f(q,t)$,

$$\mathcal{L}' = \mathcal{L} + \frac{\mathrm{d}f(q,t)}{\mathrm{d}t}, \tag{4.12}$$

then the action becomes

$$S' = \int \mathcal{L}'\,\mathrm{d}t = \int \left(\mathcal{L} + \frac{d}{\mathrm{d}t}f(q,t)\right)\mathrm{d}t = S + f(q,t)\Big|_{t_i}^{t_f}. \tag{4.13}$$

Since the variation of the action depends only on the integrand and variations vanish at the endpoints, the equations of motion (EOM) remain unchanged. Thus, adding a total derivative to the Lagrangian does not affect the classical dynamics.

4.2 Gauge dependence of the kernel

In quantum mechanics, particularly in the path integral formulation, the transition amplitude is given by

$$\langle q_f, t_f \mid q_i, t_i\rangle = \int \mathcal{D}q\, e^{\frac{i}{\hbar}S[q]}. \tag{4.14}$$

If we modify the Lagrangian as before,

$$\mathcal{L}' = \mathcal{L} + \frac{d}{\mathrm{d}t}f(q,t), \tag{4.15}$$

the action becomes

$$S' = S + f(q,t)\Big|_{t_i}^{t_f}, \tag{4.16}$$

so the transition amplitude becomes

$$\langle q_f, t_f \mid q_i, t_i\rangle \to e^{\frac{i}{\hbar}(f(q_f,t_f)-f(q_i,t_i))}\int \mathcal{D}q\, e^{\frac{i}{\hbar}S[q]}. \tag{4.17}$$

Thus, the wavefunction or transition amplitude picks up a phase factor:

$$\boxed{\psi \to e^{\frac{i}{\hbar}f(q,t)}\psi} \tag{4.18}$$

This global (or local) phase has no classical analog but plays a significant role in the Aharonov-Bohm effect.

4.3 Phase of a kernel

Lagrangian for a charged particle moving in presence of electromagnetic field:

$$L = \frac{1}{2}m\dot{x}^2 + q\dot{x}\cdot\mathbf{A}(x) - q\phi(x) \tag{4.19}$$

The action becomes (we assume electrostatic potential to be zero):

$$S[x(t)] = \int_{t_i}^{t_f}\left(\frac{1}{2}m\dot{x}^2 + q\dot{x}\cdot\mathbf{A}(x)\right)\mathrm{d}t \tag{4.20}$$

Focus on the 2nd term in the action integral:

$$S_A = q\int_{t_i}^{t_f}\dot{x}(t)\cdot\mathbf{A}(x(t))\mathrm{d}t \tag{4.21}$$

Use:

$$\dot{x}(t)\mathrm{d}t = \mathrm{d}x(t) \tag{4.22}$$

And the term becomes:

$$S_A = q \int_{x_i}^{x_f} \mathbf{A}(x) \cdot \mathrm{d}x \tag{4.23}$$

So, the phase factor picked up by the action is:

$$\exp\left(\frac{iq}{\hbar}\int_{x_i}^{x_f} \mathbf{A}(x)\cdot dx\right) \quad \text{This is going to be important later!} \tag{4.24}$$

4.4 AB without Path Integral approach

In this section, we present a basic approach to understanding the Aharonov-Bohm (AB) effect that does not require the use of path integrals.

We begin by considering the time-dependent Schrödinger equation for a particle with charge q, in a region where the vector potential $\mathbf{A} \neq 0$ but the magnetic field $\mathbf{B} = 0$:

$$\left(\frac{1}{2m}\left(-i\hbar\nabla - \frac{q}{c}\mathbf{A}\right)\right)^2 \psi = i\hbar\partial_t\psi \tag{4.25}$$

Here, the scalar potential is assumed to be zero. A solution to equation 4.25 can be expressed as the product of the solution for $\mathbf{A} = 0$, denoted by ψ_0, and a position-dependent phase factor. Specifically,

$$\boxed{\psi = \psi_0 e^{ig(\mathbf{r})} \quad \text{with} \quad g(\mathbf{r}) = \frac{q}{\hbar c}\int_{\mathbf{r}_0}^{\mathbf{r}} \mathrm{d}\mathbf{r}'\cdot\mathbf{A}(\mathbf{r}')} \tag{4.26}$$

To verify that the wavefunction in equation 4.26 satisfies equation 4.25, we compute the gradient of ψ:

$$\nabla(\psi_0 e^{ig(\mathbf{r})}) = (\nabla\psi_0)e^{ig(\mathbf{r})} + i(\nabla g(\mathbf{r}))\psi_0 e^{ig(\mathbf{r})} \tag{4.27}$$

Given that

$$\nabla g(\mathbf{r}) = \frac{q}{\hbar c}\mathbf{A}(\mathbf{r}),$$

this result is path-independent because

$$\boldsymbol{\nabla}\times\mathbf{A} = \mathbf{B} = 0.$$

This implies the existence of a scalar potential Φ_M such that

$$\mathbf{A} = -\nabla\Phi_M.$$

Then, using Stokes' theorem:

$$\oint \mathbf{A}(\mathbf{r})\cdot\mathrm{d}\mathbf{r} = \int_S \mathbf{B}\cdot\mathrm{d}\sigma = -\int_i^f \nabla\Phi_M\cdot\mathrm{d}\mathbf{r} = -(\Phi_M)_f + (\Phi_M)_i,$$

which depends only on the endpoints of the path.

The kinetic momentum operator is defined as

$$\mathbf{\Pi} \equiv -i\hbar\nabla - \frac{q}{c}\mathbf{A}.$$

Applying $\mathbf{\Pi}$ to ψ:

$$\mathbf{\Pi}\psi = (-i\hbar\nabla - \frac{q}{c}\mathbf{A})\psi_0 e^{ig(\mathbf{r})} \tag{4.28}$$

$$= e^{ig(\mathbf{r})}\left[-i\hbar(\nabla\psi_0) - i\hbar\left(i\frac{q}{\hbar c}\mathbf{A}\right)\psi_0 - \frac{q}{c}\mathbf{A}\psi_0\right] \tag{4.29}$$

$$= -i\hbar e^{ig(\mathbf{r})}(\nabla\psi_0) \tag{4.30}$$

To compute $\mathbf{\Pi}^2\psi$, we apply π again to the result in equation 4.30:

$$\left(-i\hbar\nabla - \frac{q}{c}\mathbf{A}\right)^2 \psi = -\hbar^2(\nabla^2\psi_0)e^{ig(\mathbf{r})} \tag{4.31}$$

Substituting this back into equation 4.25 and canceling the common phase factor $e^{ig(\mathbf{r})}$, we obtain the Schrödinger equation for ψ_0:

$$-\frac{\hbar^2}{2m}\nabla^2\psi_0 = i\hbar\partial_t\psi_0 \tag{4.32}$$

Equation 4.32 confirms that ψ_0 is indeed the solution of the Schrödinger equation when $\mathbf{A} = 0$.

This analysis demonstrates that the vector potential, even in a region with no force fields, manifests itself as a physically observable phase factor in the wavefunction.

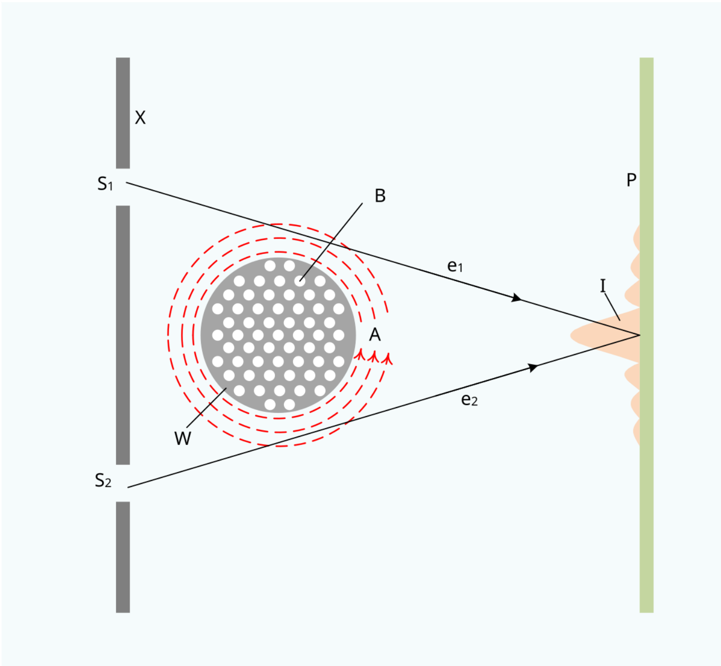


Figure 4.1: The Setup for AB effect. Credits: Wikipedia

Chapter 5

Path Integral Analysis of AB effect

In this chapter, we derive and discuss the detailed calculations as done in Gerry & Singh (1979), to determine the explicit functional form of Feynman propagator $K(\mathbf{r}'', \mathbf{r}'; \tau)$ (as given by Schulman (1971)) for the electron in AB effect. The particular essence of AB effect is that even though the electron never enters the solenoid, there are observable effects like *interference pattern* observed due to the changing magnetic flux of the infinite solenoid, despite the electron feeling no force outside ($\mathbf{B} = 0$).

The paper discusses the AB effect by introducing the singularity associated with the solenoid as a periodic constraint (Inomata & Singh (1978b)) in the path integral.

5.1 Path integrals with a periodic constraint

As discussed in the preceding chapters, the Feynmann path-integral formulation of quantum mechanics asserts that the propagator is given by

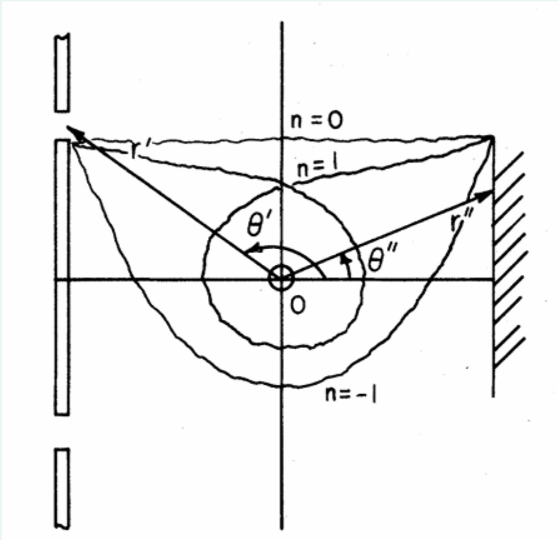
$$K(\mathbf{r}'', \mathbf{r}'; \tau) = \int \mathcal{D}[\mathbf{r}(t)] \; e^{\frac{i}{\hbar} S(\mathbf{r}'', \mathbf{r}')} \tag{5.1}$$

where the symbol $\mathcal{D}\mathbf{r}(t)$ means the integrations are to be taken over all possible paths from $\mathbf{r}' = \mathbf{r}(0)$ to $\mathbf{r}'' = \mathbf{r}(\tau)$. $S(\mathbf{r}'', \mathbf{r}')$ is the classical action given by

$$S(\mathbf{r}'', \mathbf{r}') = \int_0^\tau \mathcal{L}(\mathbf{r}, \dot{\mathbf{r}}) \mathrm{d}t \tag{5.2}$$

Construction

In these calculations we shall confine ourselves to path integrations over 2-D plane where $\mathbf{r} = (r, \theta)$. The origin of our set of polar coordinates is centered on the infinitely thin solenoid. In order to make it easier to visualize the topological constraint, we consider an idealized *string* commencing at $\mathbf{r}'$ and terminating at $\mathbf{r}''$ by stipulating that a possible configuration of the string is given by a path of the random walk from $\mathbf{r}'$ to $\mathbf{r}''$. If there are no singularities (or *poles* on the plane) any configuration of the string can be continuously changed (deformed) into other[a] without any obstruction. To impose a periodic constraint we place a singular point in the plane (at the origin). Now visualise, the configuration of string extending from $\mathbf{r}'$ to $\mathbf{r}''$ *without* encircling the singularity cannot be changed continuosly to the configuration where it is encircling the solenoid (i.e pole). In other words, we say the former configuration is homotopically in-equivalent to the later. This topologically different configurations can be classified by the number of turns n (also called *winding number*) around the solenoid.



$$n = 0, \pm 1, \pm 2, \ldots$$

n turns clockwise (from $\mathbf{r}'$ to $\mathbf{r}''$) if $n \geq 0$, and $n+1$ turns anticlockwise if $n < 0$. If $\Theta = \arccos(\mathbf{r}' \cdot \mathbf{r}'')$ is the angle between the two position vectors, then we can write, the total phase angle traversed by the electron as,

$$\int_0^\tau \dot{\theta} \mathrm{d}t = \Theta + 2\pi n$$

[a]This is an example of *simply connected region.*A simply connected region is one in which a path that surrounds the region (in this case a path for a line integral) can be continuously deformed until it becomes a point.

To select out a path with a specific angle we can impose the constraint,

$$\int_0^\tau \dot{\theta} \mathrm{d}t = \phi$$

into the propagator. Which becomes,

$$K_\phi(\mathbf{r}'', \mathbf{r}'; \tau) = \int \delta\left(\phi - \int \dot{\theta}\, \mathrm{d}t\right) \exp\left[\frac{i}{\hbar} S(\mathbf{r}'', \mathbf{r}')\right] \mathcal{D}\mathbf{r}(t) \tag{5.3}$$

The δ function selects out the different classes of homotopic configurations. Thus, the total propagator will be given by

$$K(\mathbf{r}'', \mathbf{r}'; \tau) = \int_{-\infty}^{\infty} K_\phi \, \mathrm{d}\phi \tag{5.4}$$

Using the relation (from Fourier Transform)

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}\lambda \; \exp(i\lambda x)$$

$$\Longrightarrow \; \delta\left(\phi - \int \dot{\theta} \; \mathrm{d}t\right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}\lambda \; \exp\left[i\lambda\left(\phi - \int \dot{\theta} \; \mathrm{d}t\right)\right]$$

We get from eq 5.2,

$$K_\lambda(\mathbf{r}'', \mathbf{r}'; \tau) = \int \exp\left[\frac{i}{\hbar} \int_0^\tau [\mathcal{L}(\mathbf{r}, \dot{\mathbf{r}}) - \lambda\hbar\dot{\theta}]\mathrm{d}t\right] \mathcal{D}[\mathbf{r}(t)] \tag{5.5}$$

where we write K_ϕ as,

$$\boxed{K_\phi(\mathbf{r}'', \mathbf{r}'; \tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} K_\lambda(\mathbf{r}'', \mathbf{r}'; \tau) \exp(i\lambda\phi) \, \mathrm{d}\lambda} \tag{5.6}$$

We are thus, led to consider the path integral equation (5.6) in which the original Lagrangian $\mathcal{L}$ has been replaced by an effective $\mathcal{L}'$ containing an *angular velocity-dependent-potential*

$$\mathcal{L}' = \mathcal{L} - \lambda\hbar\dot{\theta}$$

5.2 The Aharonov-Bohm Effect

We calculate the propagator for an electron moving in the magnetic field free region outside of a solenoid. The Lagrangian for this system is

$$\mathcal{L} = \frac{1}{2}m\dot{\mathbf{r}}^2 + \frac{e}{c}\mathbf{A}\cdot\dot{\mathbf{r}} \tag{5.7}$$

We choose a vector potential of the form $\mathbf{A} = A\hat{\theta}$

$$\mathbf{A} = \frac{\Phi_B}{2\pi r}\hat{\theta}$$

where Φ_B [1] is the confined magnetic flux of the solenoid, and $r = \sqrt{x^2+y^2}$ the distance of electron from the center of solenoid (origin). It can be easily checked that this vector potential satisfies $\boldsymbol{\nabla}\times\mathbf{A} = 0$. Now following Edwards (1967), we can rewrite the second term in 5.7 as

$$\frac{e}{c}\mathbf{A}\cdot\dot{\mathbf{r}} = \frac{e}{c}(A\hat{\theta})\cdot(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta}) = \frac{\Phi_B e}{2\pi c}\dot{\theta} \tag{5.8}$$

We evaluate the path integral (5.5) with the effective Lagrangian

$$\mathcal{L}' = \frac{1}{2}\mu\dot{\mathbf{r}}^2 + (\alpha - \lambda\hbar)\dot{\theta}$$

where $\alpha = e\Phi_B/2\pi c$, μ is the mass of the electron.

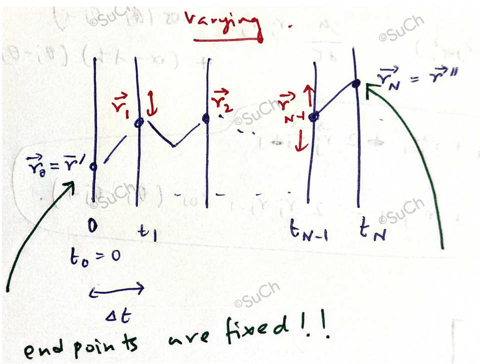
Rewriting eq 5.5, by the definition of path integral

$$K_\lambda(\mathbf{r}'', \mathbf{r}'; \tau) = \lim_{N\to\infty} A_N \int \prod_{j=1}^{N-1} \mathrm{d}\mathbf{r}_j \;\exp\left[\frac{i}{\hbar}S'[\mathbf{r}(t)]\right] \tag{5.9}$$

$$S'[\mathbf{r}(t)] = \sum_{i=1}^{N} S'(\mathbf{r}_i, \mathbf{r}_{i-1}) \tag{5.10}$$

$$= \sum_{i=1}^{N}\int_{t_{i-1}}^{t_i} \mathrm{d}t\mathcal{L}'(\mathbf{r}_i, \dot{\mathbf{r}_i}) \tag{5.11}$$

Here, we partition the entire trajectory in $N+1$ different time steps $0 = t_0, \ldots, t_N = \tau$. Then we vary the intermediate points $(\mathbf{r}_1, \ldots \mathbf{r}_{N-1})$, and keep the endpoints $(\mathbf{r}', \mathbf{r}'')$ fixed.



[1]It follows from stokes theorem on a closed circular loop enclosing an area Σ around the solenoid: $\Phi_B = \int_\Sigma \mathbf{B}\cdot\mathrm{d}\mathbf{s} = \int_\Sigma(\boldsymbol{\nabla}\times\mathbf{A})\cdot\mathrm{d}\mathbf{s} = \int_{\partial\Sigma}\mathbf{A}\cdot\mathrm{d}\mathbf{l} = A\int_0^{2\pi} r\mathrm{d}\theta = A2\pi r$

For a path integral we can approximate the function, by its values on a discrete set of paths.

$$\mathcal{L}' = \frac{1}{2}\mu\left(\frac{\mathbf{r}_i - \mathbf{r}_{i-1}}{\Delta t}\right)^2 - (\alpha - \lambda\hbar)\left(\frac{\theta_i - \theta_{i-1}}{\Delta t}\right) \equiv \mathcal{L}'(\Delta\mathbf{r}_i/\Delta t, \mathbf{r}_i) \tag{5.12}$$

where,

$$\Delta t \equiv t_i - t_{i-1} = \frac{\tau}{N}$$

So,

$$N \to \infty \leftrightarrow \Delta t \to 0$$

Therefore, partial action for a small time interval Δt may be expressed by

$$S'(\mathbf{r}_i, \mathbf{r}_{i-1}) \simeq \Delta t \mathcal{L}'(\Delta\mathbf{r}_i/\Delta t, \mathbf{r}_i)$$

such that

$$\boxed{S'[\mathbf{r}(t)] = \lim_{\Delta t \to 0} \sum_{i=1}^{N} \Delta t \mathcal{L}'(\Delta\mathbf{r}_i/\Delta t, \mathbf{r}_i)}$$

where $\mathbf{r}_i = \mathbf{r}(t_i),\ \mathbf{r}_0 = \mathbf{r}',\ \mathbf{r}_N = \mathbf{r}'',\ \theta_0 = \theta', \theta_N = \theta''$.

Now, $\Delta t\times$ Eq. 5.12 can be rewritten in polar coordinates as

$$\Delta t \mathcal{L}'(\Delta\mathbf{r}_i/\Delta t, \mathbf{r}_i) = \frac{\mu}{2\Delta t}(r_i^2 + r_{i-1}^2) + \frac{\mu}{\Delta t} r_i r_{i-1} \cos(\theta_i - \theta_{i-1}) + (\alpha - \lambda\hbar)(\theta_i - \theta_{i-1}) \tag{5.13}$$

Substituting this, we obtain

$$\exp\left[\frac{i}{\hbar}S'(\mathbf{r}_i, \mathbf{r}_{i-1})\right] = \exp\left\{\frac{i\mu}{2\Delta t\hbar}(r_i'^2 + r_{i-1}^2) - \frac{i\mu}{\hbar\Delta t} r_i r_{i-1}\left[\cos(\theta_i - \theta_{i-1}) + \frac{\hbar\Delta t}{\mu r_i r_{i-1}}\beta(\theta_i - \theta_{i-1})\right]\right\}, \tag{5.14}$$

where we set $\beta = \lambda - \alpha/\hbar$ for now. In order to take into account contributions up to fourth order in $\Delta\theta$, we employ the expansions

$$\cos(\Delta\theta) - a\Delta t\Delta\theta \simeq \cos(\Delta\theta + a\Delta t) + \frac{1}{2}a^2\Delta t^2 \tag{5.15}$$

$$\exp\left[\frac{u}{\Delta t}\cos\theta\right] = \sum_{m=-\infty}^{\infty} \exp(im\theta) I_m(u/\Delta t) \tag{5.16}$$

Derivation of 5.15

$$\begin{aligned}-\cos(\Delta\theta + a\Delta t) + \cos(\Delta\theta) &= 2\sin\left(\Delta\theta + \frac{a\Delta t}{2}\right)\sin\left(\frac{a\Delta t}{2}\right)\\ &\simeq \frac{1}{2}a^2\Delta t^2 + a\Delta t\Delta\theta \text{ (for } \Delta t \ll 1)\end{aligned}$$

Derivation of 5.16 We know modified bessel function is

$$I_\nu(z) = e^{-i\nu\pi/2} J_\nu(iz)$$

where $\nu \in \mathbb{Z}$ and $|\arg z| < \pi/2$. Now from generating function of bessel functions, we have

$$e^{iz\sin\theta} = \sum_{n=-\infty}^{\infty} J_n(z)e^{in\theta}$$

$$\implies e^{-iz\cos\theta} = \sum_{n=-\infty}^{\infty} J_n(z)e^{in\theta}e^{-in\pi/2}$$

$$\text{Substitute } -iz = \frac{u}{\Delta t}$$

$$\implies \exp\left[\frac{u}{\Delta t}\cos\theta\right] = \sum_{n=-\infty}^{\infty} e^{in\theta} I_n(u/\Delta t) \quad \square$$

The exponential 5.14 now becomes,

$$\exp\left[\frac{i}{\hbar}S(\mathbf{r}_j,\mathbf{r}_{j-1})\right] = \sum_{m_j=-\infty}^{\infty} R_{(m_j+\beta)}(r_j,r_{j-1})\exp[im_j(\theta_j-\theta_{j-1})], \tag{5.17}$$

where we have defined,

$$\boxed{R_{m_j+\beta}(r_j,r_{j-1}) = \exp\left[\frac{i\mu}{2\Delta t\hbar}(r_j^2+r_{j-1}^2)\right] I_{(m_j+\beta)}\left(\frac{-i\mu r_j r_{j-1}}{\Delta t\hbar}\right)} \tag{5.18}$$

and also made use of the asymptotic expression for the modified Bessel function

$$I_m\left(\frac{u}{\Delta t}\right) \simeq \left(\frac{\Delta t}{2\pi u}\right)^{1/2} \exp\left[\frac{u}{\Delta t} - \frac{1}{2}(m^2-\frac{1}{4})\frac{\Delta t}{u} + \mathcal{O}(\Delta t^2)\right] \tag{5.19}$$

Thus rewriting the propagator from eq. 5.9 using eq.5.17 as

$$\begin{aligned}
K_\lambda(\mathbf{r}'',\mathbf{r}';\tau) &= \lim_{N\to\infty} A_N \int \prod_{j=1}^{N-1} \mathrm{d}\mathbf{r}_j \;\exp\left[\frac{i}{\hbar}S'[\mathbf{r}(t)]\right] &(5.20)\\
&= \lim_{N\to\infty} A_N \int \prod_{j=1}^{N-1} \mathrm{d}\mathbf{r}_j \;\exp\left[\frac{i}{\hbar}\sum_{i=1}^{N} S'(\mathbf{r}_i,\mathbf{r}_{i-1})\right] &(5.21)\\
&= \lim_{N\to\infty} A_N \int \prod_{j=1}^{N-1} \mathrm{d}\mathbf{r}_j \;\prod_{i=1}^{N}\exp\left[\frac{i}{\hbar}S'(\mathbf{r}_i,\mathbf{r}_{i-1})\right] &(5.22)\\
&\text{Changing variables to polar coordinates} &(5.23)
\end{aligned}$$

$$\implies K_\lambda(r'',\theta'',r',\theta';\tau) = \lim_{N\to\infty} A_N \int\cdots\int \prod_{i=1}^{N-1}(r_i\mathrm{d}r_i\mathrm{d}\theta_i)\prod_{j=1}^{N}\sum_{m_j=-\infty}^{\infty} R_{(m_j+\beta)}(r_j,r_{j-1})\exp[im_j(\theta_j-\theta_{j-1})] \tag{5.24}$$

After interchanging the multiplication and summations, the angular integrations can be performed with the help of the orthogonality relation

$$\frac{1}{2\pi}\int_0^{2\pi}\exp[i(m'-m)\theta]\mathrm{d}\theta = \delta_{mm'} \tag{5.25}$$

The propagator K_λ becomes

$$K_\lambda(r'',\theta'',r',\theta';\tau) = \sum_{m=-\infty}^{\infty}\exp[im(\theta''-\theta')]Q_{(m+\beta)}(r'',r';\tau) \tag{5.26}$$

where

$$\boxed{Q_{(m+\beta)}(r'',r';\tau) = \lim_{N\to\infty}(2\pi)^{N-1}A_N\int\cdots\int\prod_{j=1}^{N}R_{(m_j+\beta)}(r_j,r_{j-1})\prod_{i=1}^{N-1}(r_i\mathrm{d}r_i)} \tag{5.27}$$

Now re-substituting $\beta = \lambda - \alpha/\hbar$, using definition in 5.6,

$$\begin{aligned}
K_\phi(\mathbf{r}'',\mathbf{r}';\tau) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}\lambda\; K_\lambda(\mathbf{r}'',\mathbf{r}';\tau)\exp(i\lambda\phi) &(5.28)\\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}\lambda \sum_{m=-\infty}^{\infty}\exp[im(\theta''-\theta')+i\lambda\phi]Q_{(m+\lambda-\alpha/\hbar)}(r'',r';\tau) &(5.29)
\end{aligned}$$

Making the change of variables $\lambda \to \lambda - m + \alpha/\hbar$, we have

$$K_\phi(\mathbf{r}'',\mathbf{r}';\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}\lambda\sum_{m=-\infty}^{\infty}\exp\left[im(\theta''-\theta'-\phi)+i\left(\lambda+\frac{\alpha}{\hbar}\right)\phi\right]Q_\lambda(r'',r';\tau) \tag{5.30}$$

Using the identity

$$\sum_{m=-\infty}^{\infty} \exp(im\theta) = 2\pi \sum_{n=-\infty}^{\infty} \delta(\theta + 2\pi n),$$

we get

$$K_{\phi}(\mathbf{r}'', \mathbf{r}'; \tau) = \sum_{n=-\infty}^{\infty} \delta(\theta'' - \theta' - \phi + 2\pi n) \int_{-\infty}^{\infty} \mathrm{d}\lambda \ \exp\left[i\left(\lambda + \frac{\alpha}{\hbar}\right)\phi\right] Q_{\lambda}(r'', r'; \tau). \tag{3.13}$$

The δ function is selecting out homotopic paths with $\phi = \theta'' - \theta' + 2n\pi$ if we interpret n as the winding number. Recall, the total propagator is

$$\boxed{K(\mathbf{r}'', \mathbf{r}'; \tau) = \int_{-\infty}^{\infty} K_{\phi} \mathrm{d}\phi = \sum_{n=-\infty}^{\infty} K_n(\mathbf{r}'', \mathbf{r}'; \tau),}$$

where

$$\begin{aligned} K_n(\mathbf{r}'', \mathbf{r}'; \tau) = \exp\left[\frac{i\alpha}{\hbar}(\theta'' - \theta' + 2\pi n)\right] \\ \times \int_{-\infty}^{\infty} \mathrm{d}\lambda \ \exp\left[i\lambda(\theta'' - \theta' + 2\pi n)\right] Q_{\lambda} \end{aligned} \tag{5.31}$$

Further,

$$K(\mathbf{r}'', \mathbf{r}'; \tau) = \sum_{n=-\infty}^{\infty} K_n(\mathbf{r}'', \mathbf{r}'; \tau) \tag{5.32}$$

$$= \exp\left[\frac{i\alpha}{\hbar}(\theta'' - \theta')\right] \sum_{n=-\infty}^{\infty} \exp\left[\frac{i\alpha}{\hbar} 2\pi n\right] \tilde{K}_n \tag{5.33}$$

where,

$$\boxed{\tilde{K}_n = \int_{-\infty}^{\infty} \mathrm{d}\lambda \ \exp\left[i\lambda(\theta'' - \theta' + 2\pi n)\right] Q_{\lambda}} \tag{5.34}$$

The overall angle-dependent phase factor in (5.33) can be removed if we use

$$\boxed{\psi(\mathbf{r}'', \tau) = \int K(\mathbf{r}'', \mathbf{r}'; \tau)\psi(\mathbf{r}', 0)\mathrm{d}\mathbf{r}'}, \tag{5.35}$$

and define the new wave functions and propagator (gauge transformations) as

$$\psi_{\text{new}}(\mathbf{r}, \tau) = \exp\left(\frac{i\alpha}{\hbar}\theta\right) \psi_{\text{old}}(\mathbf{r}, \tau),$$

$$K_{\text{new}}(\mathbf{r}'', \mathbf{r}'; \tau) = \exp\left[-\frac{i\alpha}{\hbar}(\theta'' - \theta')\right] K_{\text{old}}(\mathbf{r}'', \mathbf{r}'; \tau).$$

We can finally then write our propagator as

$$\boxed{K(\mathbf{r}'', \mathbf{r}'; \tau) = \sum_{n=-\infty}^{\infty} \exp\left[\frac{ine\Phi}{c\hbar}\right] \tilde{K}_n} \tag{5.36}$$

This is the form as conjectured by Schulman (1971). We evaluate eq (5.27) following Inomata & Singh (1978b) as,

$$Q_{\lambda}(\mathbf{r}'', \mathbf{r}'; \tau) = \left(\frac{\mu}{2\pi i\hbar\tau}\right) \exp\left[\frac{i\mu}{2\hbar\tau}(r''^2 + r'^2)\right] \times \delta\left(\lambda - \frac{2eg}{\hbar c}\right) I_{\lambda}\left(\frac{-i\mu r'' r'}{\hbar\tau}\right), \tag{5.37}$$

where the normalization constant $A_N = (\mu/2\pi i\hbar\Delta t)^N$. Equation 5.36 now becomes

$$\tilde{K}_n(\mathbf{r}'', \mathbf{r}'; \tau) = \left(\frac{\mu}{2\pi i\hbar\tau}\right) \exp\left[\frac{i\mu}{2\hbar\tau}(r''^2 + r'^2)\right] \\ \times \int_{-\infty}^{\infty} \mathrm{d}\lambda\ \exp[i\lambda(\theta'' - \theta' + 2\pi n)] I_\lambda \left(\frac{-i\mu r'' r'}{\hbar\tau}\right) \tag{5.38}$$

5.3 Interference!

The Feynman propagator is given by a sum over all possible homotopic classes which correspond to the winding number. For the two slits that we have, the propagators from the two slits add up due to superposition and then we have to take the sum over all those homotopic classes (5.32). i.e the propagator is given by

$$\boxed{K(\mathbf{r}'',\ \mathbf{r}'';\ \tau) = \sum_n K_n^{(1)}(\mathbf{r}',\ \mathbf{r}'';\ \tau) + \sum_n K_n^{(2)}(\mathbf{r}',\ \mathbf{r}'';\ \tau)} \tag{5.39}$$

However, we would consider only the cases for which the winding number is 0,-1. This is because for the other classes of configuration- they contribute to paths that are exotic from a semi-classical point of view and their probability amplitude is negligible.

Hence, $K_0^{(1)}(\mathbf{r}',\ \mathbf{r}'';\ \tau) \equiv K^{(1)}(\mathbf{r}',\ \mathbf{r}'';\ \tau)$, and $K_{-1}^{(2)}(\mathbf{r}'',\ \mathbf{r}'';\ \tau) \equiv K^{(2)}(\mathbf{r}',\ \mathbf{r}'';\ \tau)$ then eqn 5.39 would be approximated as;

$$K(\mathbf{r}',\ \mathbf{r}'';\ \tau) = K^{(1)}(\mathbf{r}'',\ \mathbf{r}'';\ \tau) + K^{(2)}(\mathbf{r}'',\ \mathbf{r}'';\ \tau) \tag{5.40}$$

When solenoid is off, $\mathbf{A} = 0$ (and thus $\Phi_B = 0$) then the action is the free particle action, S_0

$$K(\mathbf{r}',\ \mathbf{r}'';\ \tau) = \int_{(1)} D[\mathbf{r}] e^{\frac{i}{\hbar}S_0} + \int_{(2)} D[\mathbf{r}] e^{\frac{i}{\hbar}S_0} = \phi_0^{(1)}(\mathbf{r}'') + \phi_0^{(2)}(\mathbf{r}'')$$

where, $\int_1 D[\mathbf{r}] e^{\frac{i}{\hbar}S_0} = \phi_0^{(1)}(\mathbf{r}'')$, $\int_2 D[\mathbf{r}] e^{\frac{i}{\hbar}S_0} = \phi_0^{(2)}(\mathbf{r}'')$

From the symmetry of the problem, $\phi_0^{(1)}(\mathbf{r}'')$ and $\phi_0^{(2)}(\mathbf{r}'')$ can differ only by a phase (*Global* $U(1)$ *freedom* [2]). Moreover, the spatial symmetry of the setup dictates that they have equal moduli.

$$\phi_0^{(1)}(\mathbf{r}'') = Ce^{\frac{i}{\hbar}\theta_1} \qquad \phi_0^{(2)}(\mathbf{r}'') = Ce^{\frac{i}{\hbar}\theta_2}$$

The interference pattern is given by the modulus squared of the propagator, *i.e* from probability,

$$\begin{aligned} \|K(\mathbf{r}'',\ \mathbf{r}'',\ \tau)\|^2 &= \left\|\phi_0^{(1)}(\mathbf{r}'') + \phi_0^{(2)}(\mathbf{r}'')\right\|^2 = \left\|\phi_0^{(1)}(\mathbf{r}'')\right\|^2 + \left\|\phi_0^{(2)}(\mathbf{r}'')\right\|^2 + 2\,\mathrm{Re}\left(\overline{\phi_0^{(1)}(\mathbf{r}'')}\phi_0^{(2)}(\mathbf{r}'')\right) \\ &= 2C^2\left(1 + \cos\left(\frac{\theta_2 - \theta_1}{\hbar}\right)\right) = 4C^2\cos^2\left(\frac{\theta_2 - \theta_1}{2\hbar}\right) \end{aligned} \tag{5.41}$$

If we now turn on the solenoid,($\mathbf{A} \neq 0$) then the system is described by the following action (where q is the charge of the electron)

$$S_{tot} = S_0 + q\int A^i \mathrm{d}x_i$$

[2]Addition of a global phase $e^{i\alpha}$ to $\psi' \to \psi e^{i\alpha}$ leaves all physical predictions unchanged

The propagator now becomes,

$$
\begin{aligned}
K(\mathbf{r}',\ \mathbf{r}'';\ \tau) &= \int_{(1)} \mathcal{D}[\mathbf{r}(t)] \exp\left(\frac{\iota}{\hbar}S_0 + \frac{\iota}{\hbar} q \int_{(1)} A^i \mathrm{d}x_i\right) + \int_{(2)} \mathcal{D}[\mathbf{r}(t)] \exp\left(\frac{\iota}{\hbar}S_0 + \frac{\iota}{\hbar} q \int_{(2)} A^i \mathrm{d}x_i\right) \\
&= e^{\frac{\iota}{\hbar} q \int_{(1)} A^i \mathrm{d}x_i} \phi_0^{(1)}(\mathbf{r}'') + e^{\frac{\iota}{\hbar} q \int_{(2)} A^i \mathrm{d}x_i} \phi_0^{(2)}(\mathbf{r}'')
\end{aligned}
$$

In a given homotopy class, that is, for a specific n we have $\int A^i \mathrm{d}x_i$ same for every trajectory, as we have from stokes theorem that

$$
\oint_{1-2} A^i \mathrm{d}x_i = \int\int_{\Sigma} (\boldsymbol{\nabla} \times \mathbf{A}) \cdot \mathrm{d}\mathbf{s} = \int\int_{\Sigma} \mathbf{B} \cdot \mathrm{d}\mathbf{s} = \Phi_B
$$

which is the flux and it is constant for any loop under the same homotopy class.

The phase diff is

$$
\Delta\varphi = \frac{q}{\hbar}\left(\int_{(1)} A^i dx_i - \int_{(2)} A^i dx_i\right) = \frac{q}{\hbar}\oint_{1-2} A^i \mathrm{d}x_i = \frac{q}{\hbar}\Phi_B
$$

So we have

$$
K(\mathbf{r}',\ \mathbf{r}'';\ \tau) = e^{\frac{\iota}{\hbar} q \int_{(1)} A^i \mathrm{d}x_i} \left(\phi_0^{(1)}(\mathbf{r}'') + e^{\frac{-\iota q}{\hbar}\Phi_B} \phi_0^{(2)}(\mathbf{r}'')\right) \tag{5.42}
$$

and the corresponding probablility is

$$
\begin{aligned}
\|K(\mathbf{r}',\ \mathbf{r}'';\ \tau)\|^2 &= \left\|\phi_0^{(1)}(\mathbf{r}'') + e^{\frac{-\iota q}{\hbar}\Phi_B} \phi_0^{(2)}(\mathbf{r}'')\right\|^2 \qquad (5.43) \\
&= \left\|\phi_0^1(\mathbf{r}'')\right\|^2 + \left\|\phi_0^{(2)}(\mathbf{r}'')\right\|^2 + 2\,\mathrm{Re}\left(\overline{\phi_0^{(1)}(\mathbf{r}'')} e^{\frac{-\iota q}{\hbar}\Phi_B} \phi_0^{(2)}(\mathbf{r}'')\right) \\
&= 2C^2\left(1 + \cos\left(\frac{\theta_2 - \theta_1}{\hbar} - \frac{q}{\hbar}\Phi_B\right)\right) = 4C^2\cos^2\left(\frac{\theta_2 - \theta_1}{2\hbar} - \frac{q}{2\hbar}\Phi_B\right)
\end{aligned}
$$

There is a phase shift as seen from the Eqns 5.41 and 5.43 and this results in the interference pattern being shifted,

$$
\frac{\theta_2 - \theta_1}{\hbar} \mapsto \frac{\theta_2 - \theta_1}{\hbar} - \frac{q}{\hbar}\Phi_B
$$

As the Φ_B varies the peak of the interference pattern are shifted and this is the effect that is observed. The affect is periodic in the flux Φ_B, with period, Φ_0 is given by

$$
\frac{q}{\hbar}\Phi_0 = 2\pi \implies \Phi_0 = \frac{2\pi\hbar}{q} \tag{5.44}
$$

Chapter 6

Conclusion

The Aharonov–Bohm effect offers profound insight into the foundations of physics, particularly in how we conceptualize dynamics. In classical mechanics, forces—derived from fields—dictate motion. However, the Aharonov–Bohm effect demonstrates that in quantum mechanics, even in regions where classical forces (such as electric or magnetic fields) vanish, the *potentials* can still have observable consequences. Specifically, the electromagnetic vector potential $\mathbf{A}$ modifies the *phase* of the quantum wavefunction, and this phase difference can lead to interference effects that are experimentally measurable.

This result implies that the electromagnetic potentials, often dismissed as mere mathematical tools in classical theory, are in fact *physically real* in quantum theory. In Feynman's path integral formulation, each possible path of a particle contributes an amplitude weighted by $e^{iS/\hbar}$, where the action S includes terms involving the potentials. Thus, it is not the force or field that alters the particle's behavior directly, but the *potential*, through its influence on the action and therefore the quantum phase.

Richard Feynman's own reflections underscore this shift in perspective. He lamented that his early education emphasized fields and forces over potentials. Yet in quantum mechanics, particularly in gauge theories and the path integral approach, the potentials are not just convenient—they are *fundamental*. The Aharonov–Bohm effect thus not only illustrates a uniquely quantum phenomenon but also justifies the Lagrangian (and action-based) viewpoint as more fundamental than the Newtonian force-based approach. Our study can be further extended to test *Born Rule* in a triple slit setup as proposed by Sinha et al. (2009).

In conclusion, the Aharonov-Bohm effect challenges the classical notion that forces alone are sufficient to describe physical reality. It elevates the role of potentials—and the geometric and topological structures they imply—to a central position in modern physics, affirming that *the phase of the wavefunction, governed by the action and the potentials, carries the true physical content in quantum mechanics.*

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