Tutorial-4

① Gradient

② $\int_{(1)}^{(2)} \vec{\nabla}\phi \cdot d\vec{r} = \phi(2) - \phi(1)$

$\vec{\nabla}\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$

$= (2x+4y)\hat{i} + (4x+2z^2)\hat{j} + (4yz)\hat{k}$

$d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$

$\therefore \vec{\nabla}\phi \cdot d\vec{r} = (2x+4y)dx + (4x+2z^2)dy + 2z\,dz$

$= \int_{x=0}^{x=1} (2x+4y)dx + \int_{y=0}^{y=1} (4x+2z^2)dy + \int_{z=0}^{z=1} 4yz\,dz$

$= \int_0^1 6x\,dx + \int_0^1 (4y+2y^4)\,dy$

$= 3x^2\Big|_0^1 + \left(2y^2 + \frac{2y^5}{5}\right)_0^1$

$= 3 + 2 + \frac{2}{5}$

$= 5 + \frac{2}{5}$

$$\int_0^1 4\sqrt{z}\cdot z\cdot dz \qquad 1+1/2$$

$$= \int_0^1 4z^{3/2}\cdot dz \qquad \frac{3}{2}+1$$

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$$= \left[4\times\frac{2}{5}z^{5/2}\right]_0^1$$

$$= \frac{8}{5}$$

LHS:

$$3+2+\frac{2}{5}+\frac{8}{5} = 3+2+\frac{10}{5}2 = \boxed{7}$$

RHS

$$\phi(2) - \phi(1)$$

$$\phi(1,1,1) - \phi(0,0,0)$$

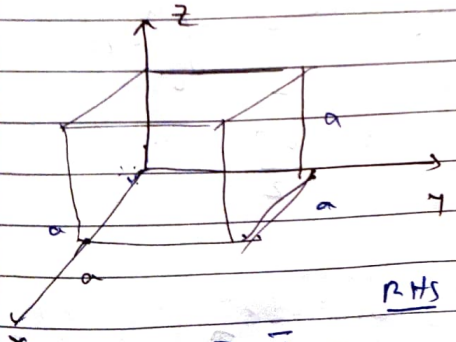
$$= 1+4+2 - 0$$

$$= 7.$$

LHS = RHS (Proved)

$$\vec{v} = xy\,\hat{i} + 2yz\,\hat{j} + 3x\,\hat{k}$$

Div. Theorem over cube of length (a) (assuming along coord. axes)



$$\oint_{\text{cube}} \vec{v}\cdot d\vec{s} = \int_V (\vec{\nabla}\cdot\vec{v})\,dV$$

RHS

$$\vec{\nabla}\cdot\vec{v} = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(2yz) + \frac{\partial}{\partial z}(3x)$$

$$= (y + 2z) + 0$$

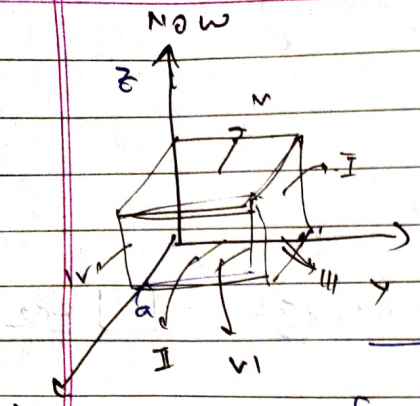
$$\therefore \int_V (\vec{\nabla}\cdot\vec{v})\,dV = \int_0^a\left[\int_0^a\left[\int_0^a (y+2z)\cdot dy\right]\cdot dz\right]\cdot dx$$

$$= \int_0^a\left[\int_0^a\left[\left(\frac{a^2}{2}+2za\right)\cdot dz\right]\cdot dx\right.$$

$$\int_0^a a\left(\frac{a^3}{2}+\frac{2a^3}{2}\right)\cdot dx = a\left(\frac{3a^3}{2}\right)$$

$$I = \frac{3a^4}{2}$$

Now



$\vec{v} = xy\,\hat{i} + 2yz\,\hat{j} + 3x\,\hat{k}$

$\oint \vec{v} \cdot d\vec{s}$ cube.

$\int_{S_I} -xy\,dy\,dz$; Now $x = 0$ $\therefore \int_{S_I} = 0$

$d\vec{s} = dy\,dz\,(-\hat{i})$

S_{II} $x = a$.

$\vec{v} = ay\,\hat{i} + 2yz\,\hat{j} + 3a\,\hat{k}$

$d\vec{s} = dy\,dz\,(\hat{i})$

$$\therefore \int_{S_{II}} ay\,dy\,dz = \int_0^a \left[\int_0^a ay\,dy\right] dz$$

$$= \frac{a^3}{2} \int_0^a dz$$

$$= \frac{a^4}{2}$$

S_{III}

$y = a$ plane.

$\vec{v} = ax\,\hat{i} + 2az\,\hat{j} + 3x\,\hat{k}$

$d\vec{s} = dx\,dz\,\hat{j}$

$$\therefore \int_{S_{III}} 2az \cdot dx\,dz = \int_0^a \left[\int_0^a 2az\,dz\right] dx$$

$$= 2a^3 \int_0^a dx = a^4$$

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S IV. $y = 0$.

$\bar{v} = 3x\,\hat{i}$

$d\bar{s} = dz\,dx(-\hat{j})$

$\bar{v} \cdot d\bar{s} = 0 \quad \therefore \int_S = 0$

S V $z = a$ plane. $\quad \bar{v}_z = 3x\,\hat{i}$.

$d\bar{s} = dx\,dy\,\hat{k}$

$$\int_{S_V} \bar{v} \cdot d\bar{s} = \int_0^a \int_0^a (3x\,dx)\,dy \quad \text{---} (1)$$

S VI. $d\bar{s} = dx\,dy(-\hat{k})$

$$\therefore \int_{S_{VI}} \bar{v} \cdot ds = -\int_0^a \int_0^a 3x\,dx \cdot dy. \quad -(II)$$

$$\therefore \oint \bar{v} \cdot d\bar{s} = \frac{a^4}{2} + a^4 + \cancel{(I)} + \cancel{(II)}$$

$$= \frac{3}{2} a^4 \quad \text{(Proved)}$$

3) $\vec{v} = xy\hat{i} + 2yz\hat{j} + 3x\hat{k}$.

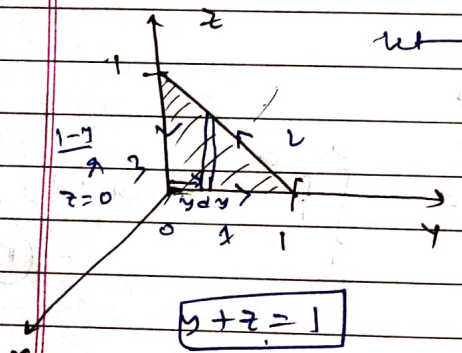
Stokes theorem

$$\oint_C \vec{v}\cdot d\vec{r} = \oint_S (\vec{\nabla}\times\vec{v})\cdot d\vec{S}$$

$$\vec{\nabla}\times\vec{v} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy & 2yz & 3x \end{vmatrix}$$

$$= i(-2y) - j(3-0) + k(-x)$$

$$= -(2y\,i + 3j + x\,k).$$



$$d\vec{S} = dz\,dy\,\hat{i}$$

$$\oint_S -2y\,dz\,dy$$

$$= \int_0^1 -2y\left(\int_0^{1-y} dz\right)\cdot dy$$

$$= \int_0^1 +2y(y-1)\cdot dy$$

$$= \int_0^1 2y^2 - 2y\cdot dy$$

$$= \left(\frac{2y^3}{3} - y^2\right)\Big|_0^1$$

$$= \frac{2}{3} - 1 = \boxed{-1/3}$$

RHS

Path 1 $\quad d\vec{r} = dy\,\hat{j}$

$$\int_1 \vec{v}\cdot d\vec{r}$$

$$= \int_0^1 2yz\,dy \qquad (\text{Now } z=0)$$

$$= 0$$

path 2

~~$d\bar{r} = dx\,\hat{i} + dy\,\hat{j}$~~ .

$d\bar{r} = dy\,\hat{j} + dz\,\hat{k}$.

$\int_2 \bar{v} \cdot d\bar{r}$

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$= \int (2yz\,dy + 3x\,dz)$.

It lies on plane. $x = 0$ $\quad y + z = 1$

$z = 1 - y$.

$= \int_1^0 2y(1-y)\,dy$

$= \int_0^1 2y(y-1)\,dy = -1/3$.

path 3

$d\bar{r} = dz\,\hat{k}$.

$\int_3 \bar{v} \cdot d\bar{r}$

Now $x = 0$ in this line.

$= \int 3x\,dz$.

$= 0$

$\therefore \oint \bar{v} \cdot dr = \int_1 \bar{v} \cdot dr + \int_2 \bar{v} \cdot dr + \int_3 \bar{v} \cdot d\bar{r}$,

$= \boxed{-1/3}$.

LHS = RHS Proved.