3/1/2022

DIFFERENTIAL EQUATIONS

Q) Solution of the D.E $\left(\frac{dy}{dx}\right)^2 - x\left(\frac{dy}{dx}\right) + y = 0$ is

A) $y = 2$ B) $y = 2x$ C) $2x - y$ D) ~~$y = 0$~~

D) $y = 2x^2 - y$

option check.

A) ~~$\frac{dy}{dx} = 2$~~ C) let $y = 2x - y$

~~$y = 2x + 2x$~~ $\frac{dy}{dx} = 2$.

$\therefore y - 2x + 2x - y = 0$ ✓

Hence A (C) $2x - y$.

DE where, y, x derivatives of y wrt x here,

$\frac{dy}{dx}$ = 1st order. $\frac{d^2y}{dx^2}$ = 2nd order. $\frac{d^3y}{dx^3}$ = 3rd order.

ie $\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$

Q) if $\frac{dy}{dx} = 2x$ & (1, 3) pass.

we can find func. cross multiply

$\int dy = \int 2x \cdot dx$ $\Rightarrow y = x^2 + C$ 1, 3

$\rightarrow 3 = 1 + C \Rightarrow C = 2$

$\therefore y = x^2 + 2$

Defn: Involves independent & dependent variables & derivatives of dependent variables is called D.E.

$y = f(x) = x^2 + 3$

x

↓ dependent on x.

↓ independent.

Partial D.E

If there are 2 or more independent variables, our concern is only ordinary DE.

Eg:-

$$\frac{\partial y}{\partial x} + \frac{\partial x}{\partial y} + \frac{\partial y}{\partial z} = 0$$

ORDINARY DE

$$\frac{d^2y}{dx^2} + 10x\left(\frac{dy}{dx}\right) + 2x^2 + 3y + 5 = 0$$

$$\Rightarrow \ddot{y} + 10x\dot{y} + 2x^2 + 3y + 5 = 0 \quad \checkmark$$

Note

Finding y (unknown fn) is called solving / Integrating the DE.

solution of DE is called its Primitive, because the DE can be regarded as, relation derived from it.

Order of DE

Defn order of Highest Differential coefficient.

$\in \mathbb{R}$

Ex → $e^{y''} + (y''')^{2020} + (\text{variable})\,y^{iv} + (\text{Also})\frac{dy}{dx} = 0$

4th order.

- $(x^2 + 7y)\frac{d^3y}{dx^3} - x\left(\frac{d^2y}{dx^2}\right)^{100} + y\left(\frac{dy}{dx}\right)^{10} = 3x$

→ 3rd order.

Degree of DE

which is expressed, or can be expressed as a polynomial in the derivatives is the degree of the highest order derivative occurring in it, after it has been expressed in a form free from Radicals & fractions

$(y')^{1/3}$ $\quad \frac{y''}{y'+3}$, $\frac{1}{y}$

so far as derivatives are concerned. Thus the DE →

$$f(x,y)\left(\frac{d^m y}{dx^m}\right)^p + g(x,y)\left(\frac{d^{m-1} y}{dx^{m-1}}\right)^q + \dots + h(x,y)\left(\frac{dy}{dx}\right)^n + k(x,y) = 0$$

p, q, ... n are non-ve Integer.

Eg:- $ax^3 + bx^2 + cx + d = 0$.

→ decreasing degree of x

अथवा {DE} → decreasing order.

Eg.. $2x(y''')^3 - y(y'')^{100} + (x^2+y)(y')^{26} - xy = 0$

Absolute term

OR,

Eg:- $(xy'')^{1/2} = 2xy + 7$.

Then square.

$xy'' = (2xy+7)^2$, deg = 2

NOW $y' + \frac{2x}{y'} = 7$. X' NO

Free from Radical & fraction.

$\Rightarrow$ $(y')^2 - 7y' + 2x = 0$

↳ Degree = 2

(अर्थ order - का highest Power → Degree.

Let $y'' = \left(y + (y')^6\right)^{1/4}$.

$\Rightarrow (y'')^4 - y - (y')^6 = 0$. order = 2, deg = 4

order of DE remains constant.

But deg. can be varied, we want minimum, Integral power.

Q) $\frac{dy}{dx} = -y + \frac{dx}{dy}$.

$\Rightarrow \left(\frac{dy}{dx}\right) = -y + \left(\frac{1}{\frac{dy}{dx}}\right)$

$\left(\frac{dy}{dx}\right)^2 = -y\left(\frac{dy}{dx}\right) + 1$ ↳ order = 1, Degree = 2

$\Rightarrow (y')^2 + y(y') - 1 = 0$

Q) $e^{y'''} - xy'' + y = 0$ ↳ $O = 3$

Degree = NOT DEFINED

Q) $\sin^{-1}\left(\frac{dy}{dx}\right) = x + y$ ↳ $O = 1$, $D = 1$

$\frac{dy}{dx} = \sin(x+y)$

Q) $\ln\left(\frac{dy}{dx}\right) = ax + by$ ↳ $O = 1$, $D = 1$

$y' = e^{ax+by}$ ✓

Q) $\frac{d^2y}{dx^2} + \sin\left(\frac{dy}{dx}\right) + 2x - y = 0$

$O = 2$

↳ D = Not Defined

As cannot be expressed as a polynomial.

$\left(e^{y'''} = 1 + y''' + \frac{(y''')^2}{\lfloor 2} + \dots \infty\right)$ ✓

multiple degree

D.E কীভাবে বসাতে হয়।

① যেখানে যেকয়টি থাকবে curve/fn

Eg:— $y = mx$; ① Const
$x^2 + y^2 + 2gx + 2fy = 0$; ② arb. const

$f(x, y, c_1, c_2 \dots c_n) = 0$

constant/ Arbitrary Constants.

② DE, Eqn in x, y & derivatives $y = mx$ of y wrt x.
(Eliminate Arbitrary constants).

---

a) $y = mx$ ①

$\frac{dy}{dx} = m$ ②

③ Eliminate m.

∴ DE is $y = (y')x$.

b) $x^2 + y^2 + 2gx + 2fy = 0$. Form DE

$2x + 2y \cdot \frac{dy}{dx} + 2g + 2f \cdot \frac{dy}{dx} = 0$ ... ①

OR ⇒ keep constant one one side.

~~$x^2 + y^2$~~ Again diff.

~~$x + y \frac{dy}{dx}$~~

$1 + \left(\frac{dy}{dx}\right)^2 + y \cdot (y'') + 0 + f \cdot d(y'') = 0$

$\Rightarrow \frac{1 + (y'')^2 + y(y'')}{(y'')} = -f$ (y'') put in eq ①

$x + y(y') + g = \left(\frac{1 + (y'')^2 + y(y'')}{(y'')}\right)(y')$

OR

$(2x + 2yy' + 2g + 2fy' = 0) \times x$.

$\Rightarrow x^2 + 2xyy' - y^2 + 2fy'(x) - 2fy = 0$.

M-2

$$\frac{x^2+y^2+2gx}{y} = -2f$$

$$\Rightarrow \frac{\left(2x+2y\cdot\frac{dy}{dx}+2g\right)(y) - (1)(N')}{y^2} = 0.$$

$$\Rightarrow y^2y' + 2xy - x^2y' = 2g(xy'-y)$$

$$\Rightarrow \frac{y^2y' + 2xy - x^2y'}{xy'-y} = 2g$$

Diff ✓

Q) Form DE of all concentric circles centre at (0,0)

So, we conclude Order = No. of Arbitrary constants.

① Curve = $x^2+y^2=r^2$

A

$$\Rightarrow 2x^0 + 2y\cdot\frac{dy}{dx} = 0$$

$$\Rightarrow \boxed{x+yy'=0}$$ Ans

Q) $y^2 = 4A(A+x)$

$$\rightarrow 2y\cdot y' = 0 + 4A$$

$$\Rightarrow \frac{y\cdot y'}{2} = A$$

$$y^2 = \frac{4}{x}y\cdot y'\left(\frac{yy'}{2}+x\right)$$

$$\rightarrow y = 2y'\cdot\left(\frac{yy'}{2}\right) + 2y'x$$

$$\rightarrow \boxed{y = (y')^2y + 2y'x}$$

Ans

Q) $y^2 = 2c(x + \sqrt{c})$ order, degree = 1

$2y \cdot \frac{dy}{dx} = 2c + 0$

No. of arbitrary constant = 1

$c = yy'$

$\therefore yx = 2xy \cdot y'(x + \sqrt{yy'})$

$\frac{y}{2y'} = x + \sqrt{yy'} \Rightarrow \sqrt{yy'} = \frac{y}{2y'} - x$

$\Rightarrow y'\sqrt{yy'} = \frac{y}{2} - xy'$

$\Rightarrow (y')^2(yy') = \left(\frac{y}{2} - xy'\right)^2$

$\Rightarrow 4y \cdot y'^3 = (y - 2xy')^2$

$\Rightarrow 4y(y')^3 - 4x^2(y')^2 + 4xyy' - y^2 = 0.$

$\therefore$ Degree = 3

Q) $ax^2 + by^2 = 1$. Form DE.

order = 2

No. of constant

Degree = 1

$2ax + 2by \cdot y' = 0$

$by y' = -ax$ — diff

$\Rightarrow \frac{yy'}{x} = -\frac{a}{b}$ $\quad \frac{(yy')'x - 1 \cdot (yy')}{x^2} = 0.$

$\Rightarrow x((y')^2 + yy'') - yy' = 0.$

Q) $y = ax^2 + bx + c$. DE. $O = ?$ $D = ?$

$y' = 2ax + b$

$y'' = 2a$ $\quad \therefore y' = (y'')x + b$

$y''' = 0$ $\quad b = y' - (y'')x$

Q) $y = c_1 e^{x} + c_2 e^{3x} + c_3 e^{4x}$

Form DE

Ans ⇒

① $y e^{-x} = c_1 + c_2 e^{2x} + c_3 e^{3x}$

② → $y' e^{-x} - e^{-x} y = 2c_2 e^{2x} + 3c_3 e^{3x}$

③ $(y' e^{-x} - e^{-x} y) e^{-2x} = 2c_2 + 3c_3 e^{3x}$

④ $-2e^{-2x}(y' e^{-x} - e^{-x} y)$
$+ e^{-2x}\left((y'') e^{-x} - e^{-x} y' + e^{-x} y - e^{-x} y'\right)$

⑤ $y'' e^{-4x} - 3y e^{-4x} - y' e^{-4x} + 3y e^{-4x} = 3c_3$

⑥ $y''' e^{-4x} - 4e^{-4x} y'' + 12 y' e^{-4x} - 3y'' e^{-4x} - y'' e^{-4x} + y' e^{-4x} + 3y' e^{-4x} - 12 y e^{-4x} = 0$

$\therefore y''' - 8y'' + y'(12 + 4 + 3) - 12y = 0.$

TRICK

$y''' - (1+3+4) y'' + (1\cdot 3 + 3\cdot 4 + 4\cdot 1) y' - (1\cdot 3\cdot 4) y = 0$

$(\alpha+\beta+\gamma)$ $\quad \Sigma\alpha\beta$ $\quad \alpha\beta\gamma$

Q) $y = c_1 e^{3x} + c_2 e^{4x}$ 2 constants → order = 2

Ans $y'' - (3+4) y' + y(3\cdot 4) = 0$

NOTE

$y = (c_1 + c_2) e^{x} + c_3$

$\because c_1 + c_2 = c$

order of DE = 1

$y = (c_1 \phi + c_2) e^{c_3} \cdot e^{x}$

$\Rightarrow y = c e^{x}$

$ax + by + c = 0$ ✗ NOT right form

$\Rightarrow y = -\frac{a}{b} \cdot x - c$

$= mx + c'$ $\Rightarrow$ 2 constants.

Find In dependent, how many?

8) $y = c_1 (\cos 2x) + c_2 \sin^2 x + c_3 \cos^2 x + c_4$

$y = c_1 \cos^2 x - c_1 \sin^2 x + c_2 \sin^2 x + c_3 \cos^2 x + c_4$

$s^2 \to 1 - c^2$

$= (1 - \cos^2 x)(c_2 - c_1) + \cos^2 x (c_1 + c_3) + c_4$

$y = 6\lambda_1 + \lambda_2 \cos^2 x$

2 arbitrary constants

General And Particular Solution.

∫ করলে constant এসে যাবে

If constant come, then called particular solution.

ELEMENTARY TYPE OF DE

- V.S (variable separable)
- Homgenous
- Linear DE

1) V.S → TYPE 1 $f(x)dx + g(y)dy = 0$

solve DE

Q) $\ln\left(\frac{dy}{dx}\right) = ax + by$.

A ① $\frac{dy}{dx} = e^{ax + by}$ (x by e^{by} both side)

Note

① $x\,dy + y\,dx = d(xy)$.

$\Rightarrow x\frac{dy}{dx} + y = \frac{d(xy)}{dx}$.

÷ to bring $\frac{dy}{dx}$

② $d\left(\frac{x}{y}\right) = \frac{y\,dx - x\,dy}{y^2}$

$\frac{d\left(\frac{x}{y}\right)}{dx} = \frac{y - x\frac{dy}{dx}}{y^2}$

Q) $y = y(x)$ represents curve in I st Quad & Let point $(1,0)$ lie on it. Tangent at any point P on curve intersects Y-Axis at Y. If PY has length 1 for all P on curve, find correct option.

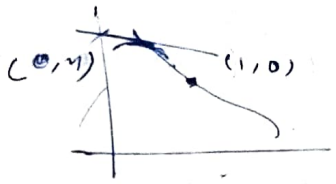
ADV 2019.

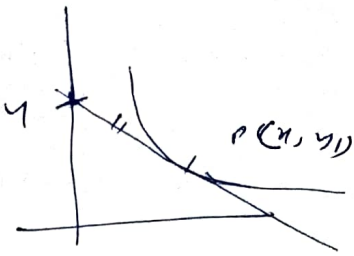
(A) $y = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right) - \sqrt{1-x^2}$

(B) $xy' - \sqrt{1-x^2} = 0$

(C) $y = -$ (A) ← option

(D) $xy' + \sqrt{1-x^2} = 0$





Eq of tangent $y - y_1 = f'(x_1)(x - x_1)$

$x = 0$ ✓ slope $x = 0$

① $y - y_1 = f'(x_1)(-x_1)$

② $(0, \quad y_1 - x_1 f'(x_1))$

③ $x_1^2 + x_1^2 f_1^2(x_1) = 1$

Dist

Generalise for all.

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{1-x^2}{x^2}$$

$$\left(\frac{dy}{dx}\right) = \pm\frac{\sqrt{1-x^2}}{x}$$

Slope +ve cannot come, $x \in [0,1]$

Suppose Graph is in 1st Quad only, & and has to pass through (1,0)

$\Rightarrow$ -ve slope, decreasing

$\Rightarrow x y' + \sqrt{1-x^2} = 0$.

Now $\frac{dy}{dx} = \frac{-\sqrt{1-x^2}}{x}$

$\Rightarrow \int dy = \int \frac{-\sqrt{1-x^2}}{x} \cdot dx$

$x = \sin\theta$

$dx = \cos\theta \, d\theta$

$\Rightarrow y = \int \frac{-\cos\theta}{\sin\theta} \cos\theta \cdot d\theta$

$= \int \frac{\sin^2\theta - 1}{\sin\theta} d\theta$

$= -\cos\theta - \ln(\operatorname{cosec}\theta - \cot\theta) + c$

Gen Soln ↑

$= -\cos\theta + \ln(\operatorname{cosec}\theta + \cot\theta)$

$y = -\sqrt{1-x^2} + \ln\left(\frac{1}{x} + \frac{\sqrt{1-x^2}}{x}\right)$

Type 2 $\frac{dy}{dx} = f(ax+by+c)$

Subs.
$ax+by+c = t$
then T-1.

Q) $\frac{dy}{dx} = \sin(x+y) + \cos(x+y)$.

Sol ① $x+y = t$

② $1 + \frac{dy}{dx} = \frac{dt}{dx}$

③ $\therefore \frac{dt}{dx} - 1 = \sin t + \cos t$

$\Rightarrow \frac{dt}{dx} = 1 + \sin t + \cos t$

$\Rightarrow \int \frac{dt}{1+\sin t + \cos t} = \int d$

TYPE-3 Exact differential Product

1) $x\,dy + y\,dx = d(xy)$

2) $\dfrac{x\,dy - y\,dx}{x^2} = d\left(\dfrac{y}{x}\right)$ $\quad \dfrac{N'D - D'N}{D^2}$ ✓

3) $\dfrac{y\,dx - x\,dy}{y^2} = d\left(\dfrac{x}{y}\right)$

4) $\dfrac{dx + dy}{x+y} = d\left(\ln(x+y)\right)$ $\quad \dfrac{f'}{f}$

5) $\frac{y\,dx - x\,dy}{x^2+y^2} = d\left(\tan^{-1}\frac{x}{y}\right)$

$\frac{y\,dx - x\,dy}{y^2} = d\left(\frac{x}{y}\right)$

$\frac{}{\left(\frac{x}{y}\right)^2+1} = \frac{}{\left(\frac{x}{y}\right)^2+1} = \int \frac{dz}{z^2+1}$

6) $\frac{x\,dx + y\,dy}{x^2+y^2} = d\left(\ln\sqrt{x^2+y^2}\right)$

$2x\,dx + 2y\,dy = dt$

$\frac{1}{2}\frac{dt}{t} = \frac{1}{2}\ln|t|$

$= \ln|\sqrt{t}|$

7) $\frac{x\,dx + y\,dy}{\sqrt{x^2+y^2}} = d\left(\sqrt{x^2+y^2}\right)$

$\frac{1}{2}\frac{dt}{\sqrt{t}} = d(\sqrt{t})$

8) $\frac{dy}{dx} = \frac{4x-3y}{3x-2y}$

$\Rightarrow 3x\,dy - 2y\,dy = 4x\,dx - 3y\,dx$

$d(2x^2)$ $\quad d(y^2)$

$\Rightarrow 3x\,dy - 4x\,dx = 2y\,dy - 3y\,dx$

$3\,d(xy)$

$\int$ $3\,d(xy) - d(y^2) - d(2x^2) = 0$

$3xy - y^2 - 2x^2 + c = 0$

9) $\frac{dy}{dx} = \frac{x-2y+5}{3x+3y-1}$ Linear / Linear

$\Rightarrow 2x\,dy + 3y\,dy - dy = x\,dx - 2y\,dx + 5\,dx$

$\Rightarrow 2x\,dy + 2y\,dx + 3y\,dy - dy$

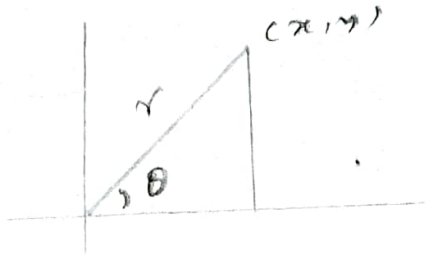
$\Rightarrow 2\,d(xy) + \frac{3}{2}\,d(y^2) - dy - \frac{1}{2}\,d(x^2) - d(5x) = 0$

$\int$ $2xy + \frac{3}{2}y^2 - y - \frac{1}{2}x^2 - 5x + c = 0$

Type-4 POLAR COORDINATES

Type a) sub. $x = r\cos\theta$, $y = r\sin\theta$.

$\sqrt{x^2+y^2}$



Type-2 sub $x = r\sec\theta$ $y = r\tan\theta$ — ①

$\sqrt{x^2-y^2}$

1) $x^2+y^2 = r^2$ $\quad \frac{y}{x} = \tan\theta$

$2x\,dx + y\,dy = r\,dr$ $\quad \frac{x\,dy - y\,dx}{x^2} = \sec^2\theta\,d\theta$

From eq ①

$x\,dx + y\,dy = r\,dr$ $\quad x\,dy - y\,dx = r^2\,d\theta$

2) $x^2-y^2 = r^2$ $\quad \frac{y}{x} = \sin\theta$

$\Rightarrow x\,dx - y\,dy = dr$ $\quad \frac{x\,dy - y\,dx}{x^2} = \cos\theta\cdot d\theta$

$= r^2\sec^2\theta\cdot\cos\theta$

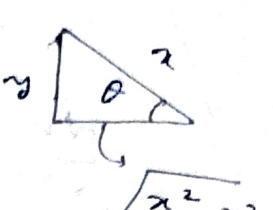
$\Rightarrow x\,dy - y\,dx = r^2\sec\theta\,d\theta$

Q) $\frac{x\,dx - y\,dy}{x\,dy - y\,dx} = \sqrt{\frac{1+x^2-y^2}{x^2-y^2}}$.

sub. $x = r\sec\theta$
$y = r\tan\theta$
$r = \sqrt{x^2-y^2}$

$= \frac{dr}{r^2 \sec\theta\, d\theta} = \sqrt{\quad}$

② $\frac{r\,dr}{r\sec\theta\, d\theta} = \sqrt{\frac{1+r^2}{r^2}}$



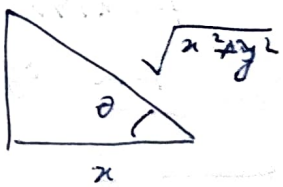
$\Rightarrow \int \frac{dr}{\sqrt{1+r^2}} = \int \sec\theta\, d\theta$

$\Rightarrow \ln\left(r + \sqrt{1+r^2}\right) = \ln(\sec\theta + \tan\theta) + C$.

$\rightarrow$ ~~$\ln(\sec\theta + \sqrt{1+r^2})$~~ =

$\Rightarrow \sqrt{x^2-y^2} + \sqrt{1+x^2-y^2} = \frac{x}{\sqrt{x^2-y^2}} + \frac{y}{\sqrt{x^2-y^2}}$.

Q) $\frac{x\,dx + y\,dy}{x\,dy - y\,dx} = \sqrt{\frac{1-x^2-y^2}{x^2+y^2}}$



$x^2+y^2 = r^2$

$\rightarrow \frac{r\,dr}{r^2\, d\theta} = \sqrt{\frac{1-r^2}{r^2}}$

$\rightarrow \int \frac{dr}{\sqrt{1-r^2}} = d\theta$

$r = \sqrt{x^2+y^2}$
$\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$\rightarrow \sin^{-1} r = \theta + C$

$\rightarrow \sin^{-1}\left(\sqrt{x^2+y^2}\right) = \tan^{-1}\left(\frac{y}{x}\right) + C$

Boards, Adv

Eqn Reducible to Homogenous Form.

Check if

1) $\frac{dy}{dx} = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$

Not II^r

$\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

and

$a_2 + b_1 \neq 0$

2) Then origin shifting.

$x = X + h$

$y = Y + k$

$\therefore dx = dX, \quad dy = dY.$

3) $\frac{dY}{dX} = \frac{dy}{dx} = $ ~~$\frac{a_1 X + b_1 Y + (a_1 h + b_1 k + c_1)}{a_2 X + b_2 Y + (a_2 h + b_2 k + c_2)}$~~

$= \frac{a_1 X + b_1 Y + (a_1 h + b_1 k + c_1)}{a_2 X + b_2 Y + (a_2 h + b_2 k + c_2)}$ ($a_1 h + b_1 k + c_1 = 0$, $a_2 h + b_2 k + c_2 = 0$)

Select (h, k) such that constant part $= 0$

Then $\frac{Y}{X} = v.$

~~Select that~~ Select $y = 0$

Ex – 1, DE.

$$\frac{dy}{dx} = \frac{2x-y+1}{6x-5y+4}$$ solve.

Sol ① $x = X+h$ $\quad dx = dX$

$y = Y+k$ $\quad dy = dY$

② $$\frac{dy}{dx} = \frac{dY}{dX} = \frac{2(X+h)-(Y+k)+1}{6(X+h)-5(Y+k)+4}$$

$$= \frac{2X-Y+(2h-k+1)}{6X-5Y+(6h-5k+4)}$$

$2h-k+1=0 \quad \times 3$

$6h-5k+4=0$

$6h-3k+3=0$

$-2k+1=0$

$1=2k \Rightarrow \boxed{k=1/2}$

$\therefore 2h+1-\frac{1}{2}=0$

$2h = -\frac{1}{2}$

$\boxed{h=-1/4}$

$\left(-\frac{1}{4}, \frac{1}{2}\right)$

③ $$\frac{dY}{dX} = \frac{2X-Y}{6X-5Y}$$

$\div X$

④ Put $\frac{Y}{X} = v \Rightarrow Y = vX$

$$\frac{dY}{dX} = v + X\frac{dv}{dX}$$

$$\Rightarrow v + X\frac{dv}{dX} = \frac{2-v}{6-5v}$$

$$X\frac{dv}{dX} = \frac{2-v}{6-5v} - v = \frac{2-v-6v+5v^2}{6-5v}$$

$$= \frac{5v^2-7v+2}{6-5v}$$

$$\Rightarrow \int \frac{(6-5v)}{5v^2-7v+2}\,dv = -\int \frac{dX}{X} = -\ln X + C$$

$$\Rightarrow \int \frac{-(6-5v)}{5v^2-7v+2}\,dv = -\ln X + C$$

$$\int \frac{-(6-5v)}{(v-1)(5v-2)}\,dv = -\ln X + C$$

$$\Rightarrow \frac{6 - 5v}{(v-1)(5v-2)} = \frac{A}{v-1} + \frac{B}{(5v-2)}$$

$v = 1$ $\quad$ $v = 2/5$

$A = \frac{-1}{3}$, $\quad B = \frac{6 - 5\times\frac{2}{5}}{\frac{2}{5} - \frac{5}{5}} = \frac{4}{\frac{-3}{5}} = \frac{-20}{3}$

$$\Rightarrow -\frac{1}{3}\ln|v-1| - \frac{20}{3}\ln|5v-2| = -\ln x$$

$$\Rightarrow -\frac{1}{3}\ln\left|\frac{y}{x} - 1\right| + \frac{20}{3}\left(\frac{1}{5}\right)\ln\left|5\frac{y}{x} - 2\right| = +\ln x$$

$$\Rightarrow -\frac{1}{3}\ln\left|\frac{y - 1/2}{x + 1/4} - 1\right| + \frac{4}{3}\ln\left|\frac{5(y - \frac{1}{2})}{x + 1/4} - 2\right| + \ln\left(x + \frac{1}{4}\right) = c$$

Q) $\frac{dy}{dx} = \frac{2(y+2)^2}{(x+y-1)^2}$ Solve.

Ans $x = X + h$, $y = Y + k$, $\therefore \frac{dY}{dX} = 2\left(\frac{Y + 2 + k}{X + Y + h + k - 1}\right)^2$

$2 + k = 0$, $k = -2$

$h - 2 - 3 = 0$, $h = 3$

$(3, -2)$

$$\Rightarrow \frac{dy}{dx} = \frac{2Y^2}{(X+Y)^2} \quad \div X^2$$

$$\Rightarrow v + X\frac{dv}{dX} = \frac{2v^2}{(1+v)^2}$$

$$\Rightarrow X\frac{dv}{dX} = \frac{2v^2 - v - 2v^2 - v^3}{(1+v)^2}$$

$$\Rightarrow \int \frac{v^2 + 2v + 1}{v^3 + v}\, dv = \int -\frac{dx}{x}$$

$$\Rightarrow \int \left(\frac{1}{v} + \frac{2v}{v^2+1}\right) dv = -\ln X + c$$

LINEAR DE

17/1/2022

A linear D.E is if dependent variable and all its differential coefficients occur in degree one only and are never multiplied Together. So the nth order linear DE eq is like.

$$a_0(x)\left(\frac{d^n y}{dx^n}\right) + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + a_2(x)\frac{d^{n-2}y}{dx^{n-2}} + \dots + a_{n-1}(x)\left(\frac{dy}{dx}\right) + a_n(x)y = g(x).$$

Q) $\left(\frac{d^2y}{dx^2}\right)^1 + \left(\frac{dy}{dx}\right)^3 + y^2 + x = 0$

Degree = 1 (of highest order)

It is also not a DE, as all diff. coeff. must be one degree, here 2nd term is cubed., 3rd term is sq.

a) $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y + x \rightarrow$ DE linear.

Q) $(y''')^1 + x(y'')^1 + x^{2021}(y')^1 + (y)^1 - 3 = 0$

→ linear DE.

Linear DE of order 1

$$a_0(x)\cdot\frac{dy}{dx} + a_1(x)\,y = g(x)$$

$$\Rightarrow \boxed{y\,P(x) + \frac{dy}{dx} = Q(x)} \Rightarrow \frac{dy}{dx} + Py = Q$$

Exact Differential

$$\frac{x\,dy - y\,dx}{x^2} = d\left(\frac{y}{x}\right)$$

$\because$ That product rule '+'

$\hookrightarrow$ Int. $\left(x^2 = \frac{y}{x} + c\right) *$

"(b वाला"

NOTE

~~I.f = Integ~~

Integrating Factor (I.f).

to get on LHS $\rightarrow$ d (something)

$$I.f = e^{\int P(x)\,dx} = e^{\int P}$$

$\hookrightarrow$ No constant of Integration used.

$$\therefore e^{\int P}\left(\frac{dy}{dx}\right) + (e^{\int P} \cdot P)y = e^{\int P} \cdot Q$$

$$\Rightarrow \frac{d}{dx}(e^{\int P} \cdot y) = e^{\int P} Q$$

Integrate both side.

$$\Rightarrow e^{\int P} y = \int (e^{\int P} Q \cdot dx)$$

$$I.F \times y = \int (I.F \times RHS) \cdot dx$$

Note

1. $\frac{dx}{dy} + P(y)x = Q(y)$

$x \leftrightarrow y$

Same. $x \cdot I.f = \int (I.f \times RHS)\,dy.$

Eqn reducible to LDE

(Bernoulli's Eqn)

$$\frac{dy}{dx} + P(x)\,y = Q(x)\cdot y^{n}$$

TRICK

RHS side fn of x^{n}

$\therefore \div y$ power

$$\left(y^{-n}\cdot\frac{dy}{dx} + P(x)\,y^{(1-n)} = Q(x) \right) \times (1-n)$$

if $y^{1-n} = z$

$$(1-n)\,y^{-n}\cdot\frac{dy}{dx} = \frac{dz}{dx}$$

$$\frac{dz}{dx} + (1-n)\,P(x)\,y^{(1-n)} = Q(x)$$

$$\Rightarrow \frac{dz}{dx} + \underbrace{(1-n)P(x)}_{f(x)} \cdot z = \underbrace{(1-n)Q(x)}_{g(x)}$$

Q) $\frac{dy}{dx} = xy + x^3y^3$

$$\frac{dy}{dx} - xy = x^3y^3$$

① $\frac{dy}{dx} - xy = x^3y^3$

$\div y^3$

② $\left(\frac{1}{y^3}\frac{dy}{dx} - x\frac{1}{y^2} = x^3\right) \times 2$

$f(x)$

③ Subs $\frac{-1}{y^2} = z$

$\frac{dz}{dx}$

$-\frac{1}{y^2} \to f' \quad \frac{2}{y^3}$

$$\frac{2}{y^3}\cdot\frac{dy}{dx} = \frac{dz}{dx}$$

$$z \cdot (2x) = 2x^3.$$

$$\Rightarrow \frac{dz}{dx} + z \cdot (2x) = 2x^3.$$

If $e^{\int 2x\,dx} = e^{x^2}$

$$(z \cdot e^{x^2}) = \int (e^{x^2})(2x)\, x^2\,dx$$

$$\int e^t \cdot (t+1-1)\,dt$$

$$e^t t - e^t + C$$

$$\Rightarrow z e^{x^2} = e^t \cdot t - e^t + C$$

DE of form $x\,dy \cdots f(xy) + y\,dx\,g(xy) = 0$

Type

$$y f(xy)\,dx + x g(xy)\,dy = 0$$

Put $xy = v$ form D.E in v and x.

Q) $(x^3y^3 + x^2y^2 + xy + 1)\,y\,dx + (x^3y^3 - x^2y^2 - xy + 1)\,x\,dy = 0$

$(\div dx)$

① diff w.r.t x.

$y + x\frac{dy}{dx} = \frac{dv}{dx}$ $\quad y = \frac{v}{x}$.

② $(v^3 + v^2 + v + 1)\left(\frac{v}{x}\right) + (v^3 - v^2 - v + 1)\left(\frac{dv}{dx} - \frac{v}{x}\right) = 0$

$\Rightarrow 2(v^2 + v)\frac{v}{x} + (v^3 - v^2 - v + 1)\frac{dv}{dx} = 0.$

$\frac{2(v^2+v)}{x}\,v = -\frac{dv}{dx}(v^3 - v^2 - v + 1)$

$\Rightarrow \frac{2(v^2+v)\,v}{(v^3 - v^2 - v + 1)}\,dx = -\frac{dx}{x} = \int \frac{v^3 - v^2 - v + 1}{2v^2(v+1)}\,dv$

$= \frac{1}{2}\int \frac{v^2(v-1) - 1(v-1)}{v^2(v+1)}$

$= \frac{1}{2}\int \frac{(v^2-1)(v-1)}{v^2(v+1)}$

$= \frac{1}{2}\int \frac{(v-1)(v+1)(v-1)}{v^2(v+1)}$

$= \frac{1}{2}\int \frac{v^2 - 2v + 1}{v^2}$

$= \frac{1}{2}\int 1 - \frac{2}{v} + \frac{1}{v^2} = \int -2\frac{dx}{x}$

$\Rightarrow v + 2\ln x - \frac{1}{v} + 2\ln x = c$

$= \frac{1}{2}\int \frac{(v+1)(v^2 - v + 1) - v(v+1)}{v^2(v+1)}$

$= \frac{1}{2}\int 1 + \frac{1}{v^2}\,dv$

$= \frac{1}{2}\left(v - \frac{1}{v}\right)$