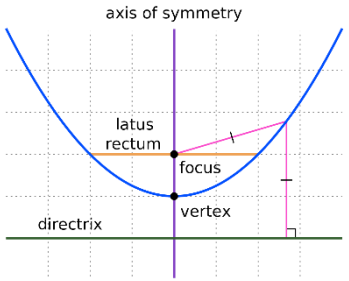
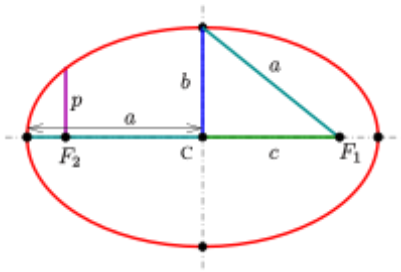
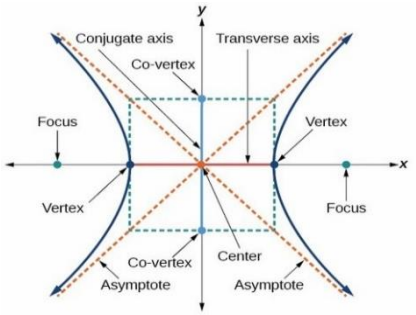
Conic Sections Formulæ

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FIGURES	GENERAL EQUATION	EQUATION OF TANGENT	EQUATION OF NORMAL	POSITION OF A POINT	POSITION OF A LINE
 **Parabola** $e = 1$	• $y^2 = 4ax$, $V(0,0), S(a,0), D \equiv x + a = 0$ • $x^2 = 4ay$ $V(0,0), S(0,a), D \equiv y = -a$ Standard Equation: $(y-k)^2 = 4a(x-h)^2$ Parametric Coord: $P(at^2, 2at)$ Length of L.R= $4a$ Focal Chords: with ends(t1&t2) $t_1.t_2 = -1$ $l = a\left(t + \frac{1}{t}\right)^2 = 4a\,\mathrm{cosec}^2\theta$ Angle made with +X axis *where* $t_1 = (at^2, 2at)$ & $t_2 = (\frac{a}{t^2}, \frac{-2a}{t})$ It's Director Circle is its **Directrix** only.	1. Point Form at (x_1,y_1) $T \equiv yy_1 - 2a(x + x_1) = 0$ 2. Parametric at $(at^2, 2at)$ $ty = x + at^2$ 3. Slope Form for L: y=mx + c *For* $y^2 = 4ax$: $y = mx + \frac{a}{m}$ *For* $x^2 = 4ay$: $y = mx - am^2$ *where* $m_T = \frac{1}{t}$ Condition for tangency: $c = \frac{a}{m}$ Pair of Tangents from a given ext. point: $S.S_1 = T^2$	1. Point Form at (x_1,y_1) $\frac{y - y_1}{x - x_1} = \frac{-y_1}{2a}$ 2. Parametric at $(at^2, 2at)$ $y + tx = 2at + at^3$ 3. Slope Form for L: y=mx + c *For* $y^2 = 4ax$: $y = mx - 2am - am^3$ *where* $m_N = -t$ Point of intersection of normals at t1 & t2 are $[2a\,(t_1^2 + t_2^2 + t_1t_2), -at_1t_2\,(t_1 + t_2)]$	For a S(x,y) $S \equiv (y^2 - 4ax)$ If • $S_1 > 0$ OUTSIDE • $S_1 = 0$ ON • $S_1 < 0$ INSIDE Chord with given Mid Pt.(x_1,y_1): $T = S_1$ Chord of Contact: $T = 0$	L: y=mx + c and P: $y^2 = 4ax$ Homogenous Eqn: < *not reqd* :) > *where* $D = 16a(a - mc)$ If • $D > 0 \rightarrow c < \frac{a}{m}$ INTERSECTING • $D = 0 \rightarrow c = \frac{a}{m}$ TANGENT • $D < 0 \rightarrow c > \frac{a}{m}$ NON-INTERSECTING Length of the chord intercepted by the parabola on the line L is: $(\frac{4}{m^2})\sqrt{a(1+m^2)(a-mc)}$
 **Ellipse** $0 < e < 1$	• $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a > b$ $V(\pm a, 0), S(\pm ae, 0), D \equiv x = \frac{\pm a}{e}$ $b^2 = a^2(1 - e^2)$; Length of L.R= $\frac{2b^2}{a}$ Major AX: y=0 ; Minor AX: x=0; $SP + S'P = 2a$ • $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a < b$ $V(0, \pm b), S(0, \pm be), D \equiv x = \frac{\pm b}{e}$ $a^2 = b^2(1 - e^2)$; Length of L.R= $\frac{2a^2}{b}$ Parametric: $P(a\cos\theta, b\sin\theta)$ Director Circle: $x^2 + y^2 = a^2 + b^2$	1.Point Form at (x_1,y_1) $T \equiv \frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$ 2.Parametric at $P(\theta)$ $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$ 3.Slope Form for L: y=mx + c c $y = mx \pm \sqrt{a^2m^2 + b^2}$ With pt. of contact at $(\frac{\pm a^2m}{c}, \frac{\mp b^2}{c})$	1. Point Form at (x_1,y_1) $\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2 = a^2e^2$ 2. Parametric Form $\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$ 3. Slope Form $y = mx \pm \frac{m(a^2 - b^2)}{\sqrt{a^2 + b^2m}}$	For a E(x,y) $E \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1$ $E_1 >, = or < 0$ OUT, ON, IN Chord with given Mid Pt.(x_1,y_1): $T = E_1$ Chord of Contact: $T = 0$ Pair of tangents $EE_1 = T^2$	Homogenous Eqn: < *not reqd* :) > If • $D > 0 \rightarrow a^2m^2 + b^2 > c^2$ INTERSECTING • $D = 0 \rightarrow a^2m^2 + b^2 = c^2$ TANGENT • $D < 0 \rightarrow a^2m^2 + b^2 > c^2$ NON-INTERSECTING Chord joining two points with eccentric angles α & β $\frac{x}{a}\cos\frac{\alpha+\beta}{2} + \frac{y}{b}\sin\frac{\alpha+\beta}{2} = \cos\frac{\alpha-\beta}{2}$
 **Hyperbola** $e > 1$	• $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ $V(\pm a, 0), S(\pm ae, 0), D \equiv x = \frac{\pm a}{e}$ $b^2 = a^2(e_1^2 - 1)$; Length of L.R= $\frac{2b^2}{a}$ Trans. AX: y=0 ; Conj. AX: x=0; $\lvert SP - S'P\rvert = 2a$ Conjugate Hyperbola: $\frac{-x^2}{a^2} + \frac{y^2}{b^2} = 1$ $V(0, \pm b), S(0, \pm be), D \equiv x = \frac{\pm b}{e}$ $a^2 = b^2(e_2^2 - 1)$; Length of L.R= $\frac{2a^2}{b}$ Parametric: $P(a\sec\theta, b\tan\theta)$ Director Circle: $x^2 + y^2 = a^2 - b^2$ Asymptotes: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$	1.Point Form at (x_1,y_1) $T \equiv \frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$ 2.Parametric at $P(\theta)$ $\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$ 3.Slope Form for L: y=mx + c c $y = mx \pm \sqrt{a^2m^2 - b^2}$ With pt. of contact at $(\frac{\pm a^2m}{c}, \frac{\pm b^2}{c})$	4. Point Form at (x_1,y_1) $\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 - b^2 = a^2e^2$ 5. Parametric Form $\frac{ax}{\sec\theta} - \frac{by}{\tan\theta} = a^2 + b^2$ 6. Slope Form $y = mx \pm \frac{m(a^2 + b^2)}{\sqrt{a^2 - b^2m}}$	For a H(x,y) $H \equiv \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1$ $H_1 >, = or < 0$ OUT, ON, IN Chord with given Mid Pt.(x_1,y_1): $T = H_1$ Chord of Contact: $T = 0$ Pair of tangents $HH_1 = T^2$	Homogenous Eqn: < *not reqd* :) > If • $D > 0 \rightarrow a^2m^2 - b^2 > c^2$ INTERSECTING • $D = 0 \rightarrow a^2m^2 - b^2 = c^2$ TANGENT • $D < 0 \rightarrow a^2m^2 - b^2 > c^2$ NON-INTERSECTING Chord joining two points with eccentric angles α & β $\frac{x}{a}\cos\frac{\alpha-\beta}{2} - \frac{y}{b}\sin\frac{\alpha+\beta}{2} = \cos\frac{\alpha+\beta}{2}$

Conic Sections Formulæ

GENERAL EQUATION OF A CONIC: FOCAL DIRECTRIX PROPERTY:

The general equation of a conic with focus (p, q) & directrix $lx + my + n = 0$ is:

$$(l^2 + m^2)[(x-p)^2 + (y-q)^2] = e^2(lx + my + n)^2 \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

**To, find the Eq$_n$ of Common Tangent to any 2 Conics, we write their equations, and equate their constant parts.

The locus of intersection of the tangents which meet at right angles is called the Director Circle

IMPORTANT POINTS ON PARABOLA: ~

- Perpendicular distance from focus on directrix = half the latus rectum.
- General Equation: $(Dist.\,from\,axis)^2 = \pm l(\mathrm{L.R})(Dist.\ \ from\,tangent\,at\,vertex)$
- Vertex is middle point of the focus & the point of intersection of Directrix & axis.
- Two parabolas are laid to be equal if they have the same latus rectum.
- Point of intersection of the tangents at t_1 & t_2 is $[at_1\,t_2,\ a(t_1 + t_1)]$.
- Point of intersection of normals at t_1 & t_2 are $[2a\,(t_1^2 + t_2^2 + t_1t_2), -at_1t_2\,(t_1 + t_2)]$.
- If t_1 & t_2 are the ends of a focal chord of the parabola $y^2 = 4ax$ then $t_1t_2 = -1$. Hence the coord. of the extremities of a focal chord: $(at^2, 2at)$ & $(\frac{a}{t^2}, \frac{-2a}{t})$.
- From a given point A(h,k) only 3 normals can be drawn at P(h,k) by $am^3 + (2a-h)m + k = 0$ *where* $m_1 + m_2 + m_3 = 0$ & $t_1 + t_2 + t_3 = 0$.
- If the normals to the parabola $y^2 = 4ax$ at the point t_1, meets the parabola again at the point t_2, then $t_2 = -(t_1 + \frac{2}{t_1})$.
- If the normals to the parabola $y^2 = 4ax$ at the points t_1 & t_2 intersect again on the parabola at the point 't_3' then $t_1t_2 = 2$ & $t_1 = -(t_1 + t_2)$ and the line joining t_1 & t_2 passes through a fixed point (-2a, 0).

IMPORTANT POINTS ON ELLIPSE: ~

- Eccentricity is given by $e = \sqrt{1 - \frac{(minor\,axis)^2}{(major\,axis)^2}}$ and Length of Latus Rectum $= \frac{2(minor\,axis)^2}{major\,axis}$.
- General Equation: $\frac{(Dist.from\,MIN\,Ax)^2}{(Semi-Major\,Ax)^2} + \frac{(Dist.from\,MAJ\,Ax)^2}{(Semi-Minor\,Ax)^2} = 1$
- Point of intersection of the tangents at α & β is $(a\frac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}, b\frac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}})$
- The figure formed by the tangents at the extremities of latus rectum is rhombus of area $= \frac{2a^2}{e}$
- DIAMETER: The locus of the middle points of a system of parallel chords with slope 'm' of an ellipse is a straight line passing through the centre of the ellipse, called its diameter and has the equation $y = \frac{-b^2}{a^2m}x$.

Conic Sections Formulæ

- The eccentric angles of point of contact of two parallel tangents differ by π. Conversely if the difference between the eccentric angles of two points is π then the tangents at these points are parallel.
- The product of the lengths of the perpendicular segments from the foci on any tangent to the ellipse is $P_1P_2 = b^2$ and the feet of these perpendiculars Y, Y' lie on its auxiliary circle. The tangents at these feet to the auxiliary circle meet on the ordinate of P and that the locus of their point of intersection is a similar ellipse as that of the original one. Also, the lines joining centre to the feet of the perpendicular Y and focus to the point of contact of tangent are parallel.
- If the normal at any point P on the ellipse with centre C meet the major & minor axes in G & g respectively, & if CF be perpendicular upon this normal, then
  (i) PF. PG = b^2 (ii) PF. Pg = a^2 (iii) PG. Pg = SP. S'P (iv) CG. CT = CS^2 (v) locus of the midpoint of Gg is another ellipse having the same eccentricity as that of the original ellipse. [where S and S' are the foci of the ellipse and T is the point where tangent at P meet the major axis]
- The tangent & normal at a point P on the ellipse bisect the external & internal angles between the focal distances of P. This refers to the well-known reflection property of the ellipse which states that rays from one focus, are reflected through other focus & vice versa. Hence, we can deduce that the straight lines joining each focus to the foot of the perpendicular from the other focus upon the tangent at any point P meet on the normal PG and bisects it where G is the point where normal at P meets the major axis.
- The portion of the tangent to an ellipse between the point of contact & the directrix subtends a right angle at the corresponding focus.
- The circle on any focal distance as diameter touches the auxiliary circle.
- Perpendiculars from the centre upon all chords which join the ends of any perpendicular diameters of the ellipse are of constant length.
- If the tangent at the point P of a standard ellipse meets the axis in T and t and CY is the perpendicular on it from the centre then,
  (I) Tt. PY = a^2 - b^2 and (II) least value of Tt is a + b.

IMPORTANT POINTS ON HYPERBOLA: ~

- General Equation: $\frac{(Dist.from\ CONJ\ Ax)^2}{(Semi-trans\ Ax)^2} - \frac{(Dist.from\ TRANS\ Ax)^2}{(Semi-conj\ Ax)^2} = 1$
- Eccentricity is given by $e = \sqrt{1 + \frac{(Conj\ axis)^2}{(Trans\ axis)^2}}$ and Length of Latus Rectum $= \frac{2(Conj\ axis)^2}{Trans\ axis}$.
- Two hyperbolas such that transverse & conjugate axes of one hyperbola are respectively the conjugate & the transverse axes of the other are called CONJUGATE HYPERBOLAS of each other.
- I) If e1 & e2 are the eccentricities of a hyperbola and its conjugate then $e_1^{-2} + e_2^{-2} = 1$.
  II) The foci of a hyperbola and its conjugate are concyclic and form the vertices of a Square.
  III) Two hyperbolas are said to be similar if they have the same eccentricity.
- The particular kind of hyperbola in which the lengths of the transverse & conjugate axis are equal is called an **Equilateral/Rectangular Hyperbola**. Note that the eccentricity of the rectangular hyperbola is $\sqrt{2}$ and the length of its L.R is equal to its transverse or conjugate axis. Equation $xy = c^2$ & $P(t) = \left(ct, \frac{c}{t}\right)$; $m_T = -\frac{1}{t^2}$
- AUXILIARY CIRCLE: A circle described on transverse axis as diameter is called the auxiliary circle. D=2a and r=a; $x^2 + y^2 = a^2$
- Note that there are two parallel tangents having the same slope m.

Conic Sections Formulæ

- Point of intersection of the tangents at α & β is $(a\frac{\cos\frac{\alpha-\beta}{2}}{\cos\frac{\alpha+\beta}{2}}, b\frac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha+\beta}{2}})$
- The portion of the tangent between the point of contact & the directrix subtends a right angle at the corresponding focus.
- ASYMPTOTE: It is a tangent at infinity.
  Both hyperbola and its conjugate have the same asymptote.
  It will always pass through the centre and the axis will be its bisectors.
  The Asymptotes of a hyperbola are the diagonals of the rectangle formed by the Tangents at the vertex of the 2 conj. Hyperbolas
- If the angle between the asymptote of a hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is 2θ *then* $e=\sec\theta$
- Perpendicular from the foci on either asymptote meet it in the same points as the corresponding directrix & the common points of intersection lie on the auxiliary circle.
- If a hyperbola is equilateral then the conjugate hyperbola is also equilateral.

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13. RECTANGULAR HYPERBOLA :

Rectangular hyperbola referred to its asymptotes as axis of coordinates.

(a) Equation is $xy=c^2$ with parametric representation $x=ct, y=c/t, t\in R-\{0\}$.

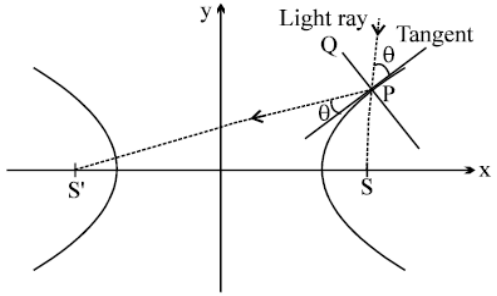
(b) Equation of a chord joining the points $P(t_1)$ & $Q(t_2)$ is $x+t_1t_2y=c(t_1+t_2)$ with slope $m=-\frac{1}{t_1t_2}$.

(c) Equation of the tangent at $P(x_1, y_1)$ is $\frac{x}{x_1}+\frac{y}{y_1}=2$ & at $P(t)$ is $\frac{x}{t}+ty=2c$.

(d) Equation of normal : $y-\frac{c}{t}=t^2(x-ct)$

(e) Chord with a given middle point as (h, k) is $kx+hy=2hk$.

The tangent & normal at any point of a hyperbola bisect the angle between the focal radii. This spells the reflection property of the hyperbola as **"An incoming light ray "** aimed towards one focus is reflected from the outer surface of the hyperbola towards the other focus. It follows that if an ellipse and a hyperbola have the same foci, they cut at right angles at any of their common point.



Note that the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the hyperbola $\frac{x^2}{a^2-k^2}-\frac{y^2}{k^2-b^2}=1(a>k>b>0)$ X are confocal and therefore orthogonal.