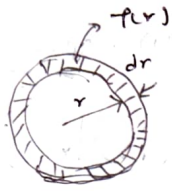
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Assignment 1

Q3) Given a star, s.t

$$\rho = \rho_c (1 - r/R)$$

- ρ_c = central Density
- $M(r = R_*) = M_*$



T.F (i) $M(r) = ?$
(ii) $M_* = ?$
(iii) $P(r) = ?$

Ans (i) Let's take an elemental spherical shell (of thickness dr) at a distance 'r' from centre of star.

It's mass $dm = \rho(r) 4\pi r^2 dr$

$$\therefore M(r) = \int_0^r dx\, \rho(x) \cdot 4\pi x^2$$

$$= 4\pi \rho_c \int_0^r dx\, x^2 (1 - x/R)$$

$$= 4\pi \rho_c \left(\frac{r^3}{3} - \frac{r^4}{4R}\right)$$

$$\therefore \boxed{M(r) = \frac{4\pi}{3} \rho_c r^3 \left(1 - \frac{3r}{4R}\right)}$$

(ii) $M_* = M(r = R_*)$

$$= \frac{1}{3} \pi R_*^3 \rho_c$$

R_* = stellar Radius, or the radius where $P = \rho = 0$

(iii) Assuming, the star to be stationary (i.e non-expanding shells/layers), and non-rotating ($\Omega = 0$), one can write the equation governing HE as:

$$\frac{dP}{dr} = -\rho(r) \frac{G M_r}{r^2}$$

Subst. $\rho(r) = \rho_c (1 - r/R)$

$$M_r = \frac{4}{3} \pi \rho_c r^3 \left(1 - \frac{3r}{4R}\right)$$

$$\frac{dP}{dr} = -\frac{4\pi G \rho_c^2}{3} r \left(1 - \frac{r}{R}\right)\left(1 - \frac{3r}{4R}\right)$$

$$= \underbrace{-\frac{4\pi}{3} G \rho_c^2}_{\wedge} \left[\left(r - \frac{r^2}{R}\right)\left(1 - \frac{3r}{4R}\right)\right]$$

$$= \wedge \left[r - \frac{3r^2}{4R} - \frac{4r^2}{4R} + \frac{3r^3}{4R^2}\right]$$

$$= \wedge \left[\text{or } \frac{3}{4}\frac{r^3}{R^2} - \frac{7}{4}\frac{r^2}{R} + r\right]$$

$$\int_{P(r)}^{0} dP = \int_r^R \wedge \left[\frac{3}{4}\frac{r^3}{R^2} - \frac{7}{4}\frac{r^2}{R} + r\right] dr$$

$$-P(r) = \wedge \left[\frac{3}{4}\frac{1}{4R^2}(R^4 - r^4) - \frac{7}{4R}\frac{1}{3}(R^3 - r^3) + \frac{(R^2 - r^2)}{2}\right]$$

Upon simplifying,

$$\boxed{P(r) = \frac{4\pi}{3} G \rho_c^2 \left[-\frac{3}{16}\frac{r^4}{R^2} + \frac{7}{12}\frac{r^3}{R} - \frac{r^2}{2} + \frac{5}{48}R^2\right]}$$

check $P(R) = 0$

Q4) Solve Lane Emden Equations analytically for $n = 0$ and $n = 1$. Using this result find the radius of the star in each case, and find a mass Radius relationship.

In the poisson's equation

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G \rho$$

If we substitute

$$\rho(r) = \rho_c \theta^n(r)$$

$$\Rightarrow P(r) = K \rho(r)^{1 + 1/n} = P_c \theta^{n+1}$$

$$r = r_n \xi \quad \left(r_n^2 = \frac{(n+1) P_c}{4\pi G \rho_c^2}\right)$$

($\xi > 0$) we can transform it into dimensionless Lane Emden Eqn.

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n \quad \cdots (*_1)$$

with solutions "polytropes of index n" $\theta_n(\xi)$

Since, $r = r_n \xi$.

Let $R = r_n \xi_1$

↑ Stellar Radius.

Finding a Generalised Mass Radius Relationship.

Recall,

$$M_r = \int_0^r 4\pi \rho\, r'^2\, dr'$$

$$= 4\pi \rho_c \int_0^r \theta^n\, r'^2\, dr'$$

subst: $r' = r_n \xi'$, $dr' = r_n\, d\xi'$

$$\therefore = 4\pi \rho_c\, r_n^3 \int_0^\xi \theta^n(\xi')\, \xi'^2\, d\xi'$$

$$= 4\pi \rho_c \frac{r^3}{\xi^3} \int_0^\xi \theta^n(\xi')\, \xi'^2\, d\xi'$$

From $(*_1)$,

$$\theta^n \xi^2 = -\frac{d}{d\xi}\left(\xi^2 \frac{d\theta}{d\xi}\right)$$

$$= 4\pi \rho_c \frac{r^3}{\xi^3} \int_0^\xi -\frac{d}{d\xi'}\left(\xi'^2 \frac{d\theta}{d\xi'}\right) d\xi'$$

$$= 4\pi \rho_c r^3 \left(\frac{-1}{\xi}\frac{d\theta}{d\xi}\right)$$

∴ At $r = R$ (stellar Radius),

Mass of star

$$\boxed{M = 4\pi \rho_c R^3 \left(\frac{-1}{\xi}\frac{d\theta}{d\xi}\right)\Bigg|_{\xi_1}}$$

(a) Solution for $n = 0$, $\theta_0 = ?$

Requirements for physical solutions.

(i) central density: $\rho(r) = \rho_c \theta^n$

$\because \rho(0) = \rho_c \Rightarrow \theta(\xi = 0) = 1$

(ii) spherical symmetry at center

$$\left(\left(\frac{dP}{dr}\right)\Big|_{r=0}\right) \rightarrow \theta' = \frac{d\theta}{d\xi} = 0 \text{ at } \xi = 0$$

→ suppresses divergent solutions of the 2nd order system.

(iii) At surface,

$P = \rho = 0 \rightarrow \theta_n = 0$ at ξ_1

first occurance of that!

Boundary conditions

- $\theta(0) = 1$, $\theta'(0) = 0$ @ $\xi = 0$ (center)
- $\theta(\xi_1) = 0$ @ $\xi = \xi_1$ (surface)

OK!

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2 \frac{d\theta}{d\xi}\right) = -1$$

$$\Rightarrow \xi^2 \frac{d\theta}{d\xi} = -\frac{\xi^3}{3} + c_1$$

$$\Rightarrow \frac{d\theta}{d\xi} = -\frac{\xi}{3} + \frac{c_1}{\xi^2}$$

Note, $c_1 = 0$, to suppress diversent solutions of θ, as $\xi \to 0$.

$$\therefore \theta = -\frac{1}{6}\xi^2 + c_2$$

Now, $\theta(0) = 1 \rightarrow c_2 = 1$

$$\therefore \boxed{\theta_0(\xi) = 1 - \frac{1}{6}\xi^2}$$

Now

$\theta_0(\xi_1) = 0 \rightarrow \xi_1 = \sqrt{6} > 0$

Check, $\frac{-1}{\xi}\frac{d\theta}{d\xi} = \frac{1}{3}$

$$\therefore \boxed{M = \frac{4}{3}\pi \rho_c R^3}$$ ∎

And Radius

$$R = \sqrt{\frac{P_c}{4\pi G \rho_c^2}} \cdot \sqrt{6}$$

$$\boxed{R = \sqrt{\frac{2}{3}\frac{P_c}{\pi G \rho_c^2}}}$$ A

In other words, $n=0$ → corresponds to a constant density sphere.

(b) Solutions for $n=1$, $\theta_1(\xi) = ?$

$$\frac{1}{\xi^2}\left(2\xi\frac{d\theta}{d\xi} + \xi^2\frac{d^2\theta}{d\xi^2}\right) = -\theta$$

$$\rightarrow \frac{d^2\theta}{d\xi^2} + \frac{2}{\xi}\frac{d\theta}{d\xi} + \theta = 0$$

Substitute, $u = \xi\theta$

$$\therefore \frac{d\theta}{d\xi} = -\frac{u}{\xi^2} + \frac{1}{\xi}\frac{du}{d\xi}$$

$$\frac{d^2\theta}{d\xi^2} = \frac{1}{\xi}\frac{d^2u}{d\xi^2} - \frac{2}{\xi^2}\frac{du}{d\xi} + \frac{2}{\xi^3}u$$

Put, subst.

$$\frac{1}{\xi}\frac{d^2u}{d\xi^2} - \frac{2}{\xi^2}\frac{du}{d\xi} + \frac{2}{\xi^3}u + \frac{2}{\xi^2}\frac{du}{d\xi} - \frac{2u}{\xi^3} + \frac{u}{\xi} = 0$$

$$\rightarrow \frac{d^2u}{d\xi^2} = -u \quad \text{SHM!}$$

$$\rightarrow u = C_1\cos(\xi) + C_2\sin(\xi)$$

$$\rightarrow \theta(\xi) = C_1\frac{\cos(\xi)}{\xi} + C_2\frac{\sin(\xi)}{\xi}$$

Recall, $\xi \to 0$ $\theta(\xi = 0) = 1$.

But $\lim_{\xi\to 0} \frac{\cos\xi}{\xi} \to \infty$

$\therefore C_1 = 0$ And $C_2 = 1$

$$\therefore \boxed{\theta_1(\xi) = \frac{\sin\xi}{\xi}}$$

Surface $\Rightarrow \theta_1(\xi_1) = 0$

$\because \xi_1 > 0$

$\Rightarrow \frac{\sin\xi_1}{\xi_1} = 0 \rightarrow \boxed{\xi_1 = \pi}$

1st Zero

check.

$$\left(\frac{-1}{\xi}\frac{d\theta}{d\xi}\right)\Big|_{\xi_1} = \frac{1}{\pi^2}$$

∴ Mass of star

$$M = 4\pi\rho_c R^3 \times \frac{1}{\pi^2}$$

$$\rightarrow \boxed{M = \frac{4}{\pi}\rho_c R^3}$$ A

And

$$R = \sqrt{\frac{2P_c}{2\cdot 4\pi G\rho_c^2}} \cdot \pi$$

$$\therefore \boxed{R = \sqrt{\frac{\pi}{2}\frac{P_c}{G\rho_c^2}}}$$ A

Q.2) $T_i = 10^9$ K

$M_{wd} = 1.4\ M_\odot$

$R = 7 \times 10^6$ m

$t_{age} = 10^{10}$ years.

Assume ρ = constant, and W.D is made of carbon.

Find current temp of star, and peak emission wavelength.

Ans we assume for simplicity

star cools by radiating it's residual thermal energy as blackbody.

Total No. of atoms in the star

$$N_a = \frac{1.4\, M_\odot}{12\, m_H} \quad (\mu = 12)$$

∵ The star is composed entirely of carbon.

∴ Total thermal energy present at the any given time

$$U_{Th} = \frac{3}{2} \times N_a K_B T_f(t)$$

(Density is constant ⇒ N_a = constant)

Since, star cools off as blackbody radiation.

$$\therefore \frac{dU_T}{dt} = -L_{star} = -4\pi R_*^2 \sigma T^4(t)$$

$$\frac{3}{2} N_a K_B \frac{dT}{dt} = -4\pi R_*^2 \sigma T^4$$

$$\int_{T_0}^{T} T^{-4}\, dT = -\frac{8\pi R_*^2 \sigma}{3 N_a K_B} \int_{t=0}^{t_{age}} dt$$

$$\left(\frac{T^{-3}}{-3}\right)\Big|_{T_0}^{T} = -\frac{8\pi R_*^2 \sigma}{3 N_a K_B}\, t_{age}$$

$$T^{-3} = \frac{8\pi R_*^2 \sigma}{N_a K_B}\, t_{age} + T_0^{-3}$$

$$T^{-3} = \frac{8\pi}{1.4} \times 12 \times m_H \times 49\times10^{12} \times 5.67 \times 10^{-8} \times \frac{10^{23}}{1.38} \times 10^{10} \times 365.25 \times 86400 + 10^{-27}$$

$1\, M_\odot = 2 \times 10^{30}$ kg

$m_H = 1.67 \times 10^{-27}$ kg.

$$\Rightarrow T^{-3} = \frac{8\pi \times 12 \times 1.67 \times 49 \times 5.67 \times 365.25 \times 86400}{1.4 \times 2 \times 1.38} \times 10^{-27+12-8-30+23+10}$$

$$\cong 1.143 \times 10^{12} \times 10^{-20} + 10^{-27}$$

$$\cong 1.143 \times 10^{-8}$$

$$\therefore \Rightarrow \text{Current Temp} = (1.143\times10^{-8})^{-1/3} = 443.93\ \text{K}$$

∴ Peak Emission wavelength

$$\lambda_m = \frac{0.00289}{443.93} = 6.5 \times 10^{-6}\ \text{m} = 6509.9\ \text{nm}$$

∴ It peaks in infrared.

Ans

$T_f = 443.93$ K

$\lambda_m = 6509.9$ nm

4) Consider infall of 1g of H, onto a star. Estimate the grav. energy release if the star is a neutron star of mass $1\, M_\odot$ and Radius = 10^4 m

Now if this H is burnt to He through pp1 chain. Energy released?

If the hydrogen falls onto the the surface of sun, what is gravitational energy release?

The mass of hydrogen will fall under this potential of NS.



$V = -\frac{GM}{r}$

∴ The total energy released by star = Total energy gained by the mass (in falling).
= Work done.

$$(\Delta E_g)_{NS} = +\int \vec{mg} \cdot d\vec{r}$$

$$= -\int_{\infty}^{R} \frac{GM}{r^2} \times m\, dr$$

$$= -GMm \int_{\infty}^{R} \frac{dr}{r^2}$$

$$= GMm \left(\frac{+1}{r}\right)_{\infty}^{R}$$

$$= \frac{GMm}{R}$$

$$= \frac{6.67 \times 10^{-11} \times 2 \times 10^{30} \times 10^{-3}}{10^4}$$

$= 13.34 \times 10^{12} = \boxed{1.334 \times 10^{13}\ J}$

Ans

For Sun

$R = 1R_\odot = 6.96 \times 10^8\ m.$

$= 6.96 \times 10^4 \times R_{NS}$ ↑ Neutron star

$\therefore (\Delta E_g)_\odot = \dfrac{1.334 \times 10^{13}}{6.96 \times 10^4}$

$\simeq \boxed{1.92 \times 10^8\ J}$

Ans

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pp1 - chain

$(_1H^1 + {}_1H^1 \longrightarrow {}_1D^2 + {}_1e^{0+} + \nu) \times 2$

$(_1D^2 + {}_1H^1 \longrightarrow {}_2He^3 + \gamma) \times 2$

$_2He^3 + {}_2He^3 \longrightarrow {}_2He^4 + 2\,{}_1H^1$

$\boxed{4\,{}_1^1H^+ + 2\,{}_{-1}^0e^- \longrightarrow {}_2^4He^{2+}} + \underbrace{\boxed{+2\nu_e + \gamma}}_{\text{Energy}}$

$\therefore$ Amt of Energy Released = Q value of reaction.

$Q \equiv \Delta E = \Delta m\, c^2$

$\Delta m\ (MeV/c^2) = [4 \times 1.00784 + 2 \times 0.00054858 + 4.00260]$

$= 2.985 \times 10^{-2}\ \dfrac{MeV}{c^2}$

$\therefore Q = 2.985 \times 10^{-2}\ MeV \times 931.5$ for $4\ {}^1H_1$

$\therefore$ For each $^1H_1$

$= \dfrac{2.985 \times 10^{-2}}{4} \times 10^6 \times 931.5\ eV$

Total H in 1g.

$= \dfrac{10^{-3}}{1.67} \times 10^{27}$

$\therefore$ Amt of Energy released for this 1g H $= \dfrac{10^{-3}}{1.67} \times 10^{27} \times \dfrac{2.985}{4} \times 10^{-2} \times 10^6 \times 931.5 \times 1.6 \times 10^{-19}\ J$

$= 0.71497 \times 10^9 \times 931.5$

$\simeq \boxed{6.66 \times 10^{11}\ J\ (Ans)}$

$\therefore$ The gravitational energy released for Neutron star is 2 orders of magnitude greater than that released by Nuclear burning of 1g Hydrogen (via pp1 chain).