Name: Sushata Chattopadhyay
Roll: 22MS077
Subj: PH4113 Assignment-10

Date: 13/11/2025
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Q1) ~~Derive~~ show $\dot{\rho} = -3\left(\frac{\dot{a}}{a}\right)[\rho + P]$

where ρ is ~~matter~~ density of universe. ($c=1$)

Ans From Friedmann Equation,

$$3\left(\frac{\dot{a}}{a}\right)^2 = 8\pi G\rho$$

differentiate it,

$$6\left(\frac{\dot{a}}{a}\right)\left[\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2\right] = 8\pi G\dot{\rho}$$

Now, from Raychaudhuri Eqn.

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}[\rho + 3P]$$

$$\Rightarrow \left(\frac{\ddot{a}}{a}\right) - \left(\frac{\dot{a}}{a}\right)^2 = -\frac{4\pi G}{3}\rho + -4\pi GP - \frac{8\pi}{3}G\rho$$

$$= -4\pi G(\rho + P)$$

$$\therefore 6\left(\frac{\dot{a}}{a}\right) \times \left[-4\pi G(\rho + P)\right]$$

$$= 8\pi G\dot{\rho}$$

$$\Rightarrow \boxed{\dot{\rho} = -3\left(\frac{\dot{a}}{a}\right)[\rho + P]}$$

Proved.

Q.2) Verify Q.(1) can also be shown by convariant conservation of space energy tensor.

$$\nabla_\mu T^{\mu\nu} = 0.$$

For FRW metric,

$$\boxed{ds^2 = -dt^2 + a^2(t)[dx^2 + dy^2 + dz^2]}$$

$u^\mu = (1, 0, 0, 0)$

We consider the stress energy tensor to be a perfect fluid.

$$\boxed{T^{\mu\nu} = (\rho + P)u^\mu u^\nu + Pg^{\mu\nu}}$$

$T^{tt} = \rho$;

$T^{xx} = T^{yy} = T^{zz} = \frac{P}{a^2}$

Connections:

$\Gamma^t_{xx} = \Gamma^t_{yy} = \Gamma^t_{zz} = a\dot{a}$

$\Gamma^x_{xt} = \Gamma^y_{yt} = \Gamma^z_{zt} = \frac{\dot{a}}{a}$

$\nabla_\mu T^{\mu\nu} = 0$

$\Rightarrow \partial_\mu(T^{\mu\nu}) + \Gamma^\mu_{\mu\alpha} T^{\alpha\nu} + \Gamma^\nu_{\mu\delta} T^{\mu\delta} = 0$

Set $\boxed{\nu = t}$

$\partial_\mu(T^{\mu t}) + \Gamma^\mu_{\mu\alpha} T^{\alpha t} + \Gamma^t_{\mu\delta} T^{\mu\delta} = 0$

since non-diagonal terms of $T^{\mu\nu} = 0$, we have

Γ^t_{ms} T^{ms}

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$\partial_t T^{tt} + \left(\Gamma^x_{xt} + \Gamma^y_{yt} + \Gamma^z_{zt}\right) T^{tt} + \Gamma^t_{xx} T^{xx} + \Gamma^t_{yy} T^{yy} + \Gamma^t_{zz} T^{zz} = 0$

$\rightarrow \dot{\rho} + 3\left(\frac{\dot{a}}{a}\right)\rho + 3\alpha \dot{a} \frac{p}{a^2}$

$\rightarrow \dot{\rho} = -3\left(\frac{\dot{a}}{a}\right)[\rho + p]$

D

Q4) Show that $T_{\mu\nu}$ for EM tensor field is:

$$T_{\mu\nu} = \frac{-2}{\sqrt{|g|}} \frac{\delta S_{EM}}{\delta g^{\mu\nu}} = g^{-\alpha\beta} F_{\mu\alpha} F_{\nu\beta} - \frac{1}{4} g_{\mu\nu} \left(g^{\rho\sigma} g^{\alpha\beta} F_{\rho\alpha} F_{\sigma\beta}\right)$$

Ans Recall, we have:

$$S_{EM} = \int d^4x \sqrt{-g} \left[\frac{-1}{4} g^{\mu\alpha} g^{\nu\beta} F_{\alpha\beta} F_{\nu\beta}\right]$$

And,

$$\delta(\sqrt{-g}) = \frac{-1}{2} \sqrt{-g}\, g_{\rho\sigma} \delta g^{\sigma\rho}$$

We can rewrite it as,

$$S_{EM} = \frac{-1}{4} \int d^4x \sqrt{-g}\, g^{\alpha\mu} g^{\beta\nu} F_{\alpha\beta} F_{\mu\nu}$$

We now vary the space-time metric, but leave the term $F_{\alpha\beta} F_{\mu\nu}$ unvaried. The variation we need is

$$\delta\left[\sqrt{-g}\, g^{\alpha\mu} g^{\beta\nu}\right] =$$

$$\frac{-1}{2} \sqrt{-g} \left(g_{\sigma\rho} \delta g^{\sigma\rho}\right) g^{\alpha\mu} g^{\beta\nu} + \sqrt{-g}\, g^{\alpha\mu} \delta g^{\beta\nu} + \sqrt{-g}\, g^{\beta\nu} \delta g^{\alpha\mu}$$

Applying this variation to action integral,

$$\delta S = \frac{-1}{4} \int d^4x \sqrt{-g} \left[\left(\delta g^{\beta\nu} F^{\alpha}{}_{\nu} F_{\alpha\beta} + \delta g^{\alpha\mu} F_{\mu}{}^{\beta} F_{\alpha\beta}\right) \rightarrow \text{same} - \frac{1}{2} F^{\mu\nu} F_{\mu\nu} g_{\sigma\rho} \delta g^{\sigma\rho}\right]$$

we cycle dummy indices in each term in order to pull out a common factor $\delta g^{\mu\nu}$

$$= \int d^4x \times \frac{-1}{2} \sqrt{-g} \left\{\left(\frac{-1}{4} g_{\mu\nu} g^{\rho\alpha} g^{\sigma\beta} F_{\rho\sigma} F_{\alpha\beta}\right) + g^{\alpha\beta} F_{\mu\alpha} F_{\nu\beta}\right\} \delta g^{\mu\nu}$$

$$\equiv \int d^4x \frac{\delta S}{\delta g^{\mu\nu}} \delta g^{\mu\nu}$$

where we also used antisymmetric property of $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$

$$= \frac{-1}{2} \int d^4x \sqrt{-g} \left[T^{EM}_{\mu\nu}\right] \delta g^{\mu\nu}$$

$\therefore$ For free EM field, stress energy tensor is

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$\therefore T_{\mu\nu} = g^{\alpha\beta} F_{\mu\alpha} F_{\nu\beta}$

$-\frac{1}{4} g_{\mu\nu} (g^{\rho\sigma} g^{\alpha\beta} F_{\rho\alpha} F_{\sigma\beta})$

D

Q5) Solve the Friedmann Equation for pressure-less dust matter ($P=0$)

plot $a(t)$, Energy density $\rho(t)$.

From Friedmann Equation,

$$\dot{\rho} = -3\left(\frac{\dot{a}}{a}\right)[\rho + P]$$

For Dust $P=0$,

$$\dot{\rho} = -3\left(\frac{\dot{a}}{a}\right)\rho$$

$$\int \frac{\dot{\rho}}{\rho} = -3 \int \left(\frac{\dot{a}}{a}\right)$$

$$\ln(\rho/\rho_0) = -3 \ln(a/a_0)$$

$$\rho = \rho_0 \left(\frac{a_0}{a}\right)^3$$

And also we know

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G \rho}{3}$$

$$\frac{\dot{a}}{a} = \left(\frac{8\pi G \rho_0 a_0^3}{3}\right)^{1/2} a^{-3/2}$$

$$\frac{da}{dt} = \left(\frac{2}{3}\xi^{3/2}\right) a^{-1/2}$$

$$\int_0^{a(t)} \sqrt{a}\, da = \xi \int_{t_0}^{t} dt$$

$$\frac{2}{3} a^{3/2} = \frac{2}{3}\xi^{2/3}(t - t_0)$$

$$\therefore \boxed{a(t) = \xi (t - t_0)^{2/3}}$$

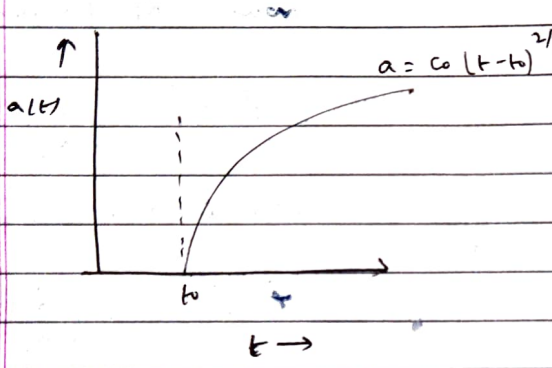
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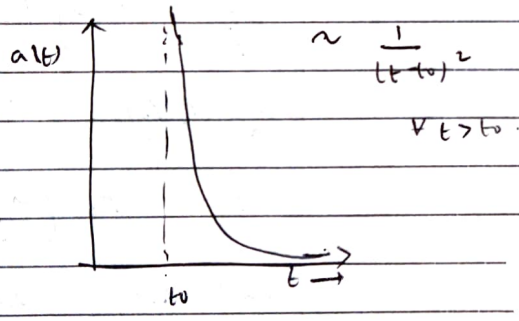
$\boxed{a(t_0) = 0}$

$$\rho = \rho_0 \left(\frac{a_0}{a}\right)^3$$

$$= \rho_0 \left(\frac{a_0}{\xi}\right)^3 \frac{1}{(t-t_0)^2}$$

$$\xi = \left[\frac{3}{2}\left(\frac{8\pi}{3} G \rho_0 a_0^3\right)^{1/2}\right]^{3/2}$$





$$\delta S_\varphi = \int d^4x \left(\frac{\delta S_\varphi}{\delta g^{\mu\nu}}\right) \delta g^{\mu\nu}$$

we have,

$$\delta(\sqrt{-g}) = \frac{-1}{2}\sqrt{-g}\, g_{\mu\nu} \delta g^{\mu\nu}$$

Now,

$$T_{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\delta S_\varphi}{\delta g^{\mu\nu}}$$

$$= \frac{\delta S_\varphi}{\left(\frac{-1}{2}\sqrt{g}\, \delta g^{\mu\nu}\right)}$$

$$= g_{\mu\nu} \frac{\delta S_\varphi}{\delta(\sqrt{-g})}$$

Q) 2.) Show that for Klein Gordon field ϕ,

$$T_{\mu\nu} = (\nabla_\mu \phi)(\nabla_\nu \phi) + g_{\mu\nu}\left[\frac{-1}{2} g^{\alpha\beta} \phi_{;\alpha} \phi_{;\beta} - V(\phi)\right]$$

The Klein Gordon scalar field in an arbitrary spacetime is given by

$$S_\varphi = \int d^4x \sqrt{-g}\left[\frac{-1}{2} g^{\mu\nu} (\nabla_\mu \phi \nabla_\nu \phi) - V(\phi)\right]$$

Varying only the spacetime metric.

Now,

$$T^{\mu\nu} := \frac{-2}{\sqrt{-g}} \left(\frac{\delta S_M}{\delta g_{\mu\nu}} \right)$$

We write action in terms of lagrangian density,

$$S[\phi] = \int d^4x \sqrt{-g}\, \mathcal{L}.$$

$$\mathcal{L} = \left(-\frac{1}{2} g^{\mu\nu} (\nabla_\mu \phi)(\nabla_\nu \phi) - V(\phi) \right)$$

$\therefore$

$$\delta S = \int d^4x \left[\mathcal{L}\, \delta\sqrt{-g} + \sqrt{-g}\, \delta\mathcal{L} \right]$$

$$= \int d^4x \left[-\frac{1}{2} \sqrt{-g}\, g_{\mu\nu} \delta g^{\mu\nu} \mathcal{L} - \frac{1}{2} \sqrt{-g} (\nabla_\mu \phi)(\nabla_\nu \phi)\, \delta g^{\mu\nu} \right]$$

$$= \int d^4x \; \frac{-1}{2} \sqrt{-g} \left[g_{\mu\nu} \mathcal{L} + (\nabla_\mu \phi)(\nabla_\nu \phi) \right] \delta g^{\mu\nu}$$

$$\equiv \int d^4x \; \frac{\delta S_M}{\delta g^{\mu\nu}} \delta g^{\mu\nu}$$

From this we identify stress energy tensor to be,

$$T_{\mu\nu} = g_{\mu\nu} \mathcal{L} + (\nabla_\mu \phi)(\nabla_\nu \phi)$$

$$= (\nabla_\mu \phi)(\nabla_\nu \phi) + g_{\mu\nu} \left\{ -\frac{1}{2} g^{\alpha\beta} \phi_{;\alpha} \phi_{;\beta} - V(\phi) \right\}$$

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