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ASSIGNMENT- 8

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Q1) Show that for interior metric

$\nabla_\mu T^{\mu\nu} = 0 \Rightarrow \frac{dP}{dr} = -(\rho+P)\frac{d\Phi}{dr}$; $\nu \in \{0,1,..4\}$

we use $c = 1$

Ans For a perfect fluid,

$$\boxed{T_{\mu\nu} = (\rho + P)\, u_\mu u_\nu + P g_{\mu\nu}}$$

ρ = Rest-frame mass density

P = Pressure (measured in Rest Frame)

u_μ = 4-velocity of fluid.

Interior Metric

$$\boxed{ds^2 = -e^{2\Phi} dt^2 + e^{2\nu} dr^2 + r^2 (d\theta^2 + \sin^2\theta\, d\phi^2)}$$

$\therefore g_{\mu\nu} = \text{diag}(-e^{2\Phi}, e^{2\nu}, r^2, r^2\sin^2\theta)$

Finding 4-velocity

$\vec{u} = \frac{d\vec{x}}{d\tau} = \frac{d}{d\tau}(t, r, \theta, \phi)$

As fluid is static (Non-rotating)

$= \left(\frac{dt}{d\tau}, 0, 0, 0\right)$

$= (u^0, 0, 0, 0)$

Now $g_{\mu\nu} u^\mu u^\nu = -c^2 = -1$

$\therefore -e^{2\Phi}(u^0)^2 = -1$

$u^0 = e^{-\Phi}$

Thus $u_0 = g_{0\nu} u^\nu = g_{00} u^0$

$= -e^{2\Phi} \cdot e^{-\Phi}$

$= -e^{\Phi}$

$\therefore$

$$\boxed{u^\mu = (e^{-\Phi}, 0, 0, 0)}$$

$\therefore$

$$\boxed{u_\mu = (-e^{\Phi}, 0, 0, 0)}$$

Now

$T^{\mu\nu} = (\rho+P) u^\mu u^\nu + P g^{\mu\nu}$

$\nabla_\mu T^{\mu\nu} = 0$

$\Rightarrow$ (is required for fluid to be in hydrostatic equilibrium).

$\nabla_\mu (\rho + P) u^\mu u^\nu + P \nabla_\mu g^{\mu\nu} + (\nabla_\mu P) g^{\mu\nu} = 0$ ($\nabla_\mu g^{\mu\nu} = 0$)

$\partial_\mu [(\rho+P) u^\mu u^\nu]$

$+ (\rho+P)\, \Gamma^\mu_{\alpha\mu} u^\alpha u^\nu$

$+ (\rho+P)\, \Gamma^\nu_{\alpha\mu} u^\mu u^\alpha$

$+ (\partial_\mu P) g^{\mu\nu} = 0$ ... (*)

We have,

$\Gamma^t_{rt} = \frac{d\Phi}{dr}$

$\Gamma^r_{tt} = e^{2(\Phi-\nu)} \frac{d\Phi}{dr}$

~~Because we want a derivative wrt 'r'~~

$$T^{\mu\nu} = \begin{bmatrix} \rho & 0 & 0 \\ 0 & P & 0 \\ 0 & 0 & P \end{bmatrix}$$

From eq (*)

$\Rightarrow \partial_t [(\rho+P) u^t u^\nu]$

$+ (\rho+P)\, \Gamma^\mu_{t\mu} u^t u^\nu$

$+ (\rho+P)\, \Gamma^\nu_{\alpha t\mu} u^t u^\mu$

$+ (\partial_\mu P) g^{\mu\nu} = 0$

(*) From eq (i) Note $\eta^{\sigma\mu} = g^{\sigma\mu} + \mathcal{O}(\epsilon)$ can also be written

$\therefore g^{\sigma\mu} = \eta^{\sigma\mu} - \epsilon h_{\alpha\beta}(g^{\alpha\mu} + \mathcal{O}(\epsilon))$

$\Rightarrow \boxed{g^{\sigma\mu} \approx \eta^{\sigma\mu} - \epsilon h^{\sigma\mu}} + \mathcal{O}(\epsilon^2)(g^{\sigma\beta} + \mathcal{O}(\epsilon)) + \mathcal{O}(\epsilon^2)$ □

Proved

Subst. $\nu = \nu$.

$0 + 0 + (\rho + P)\Gamma^{r}_{tr} u^t u^r + (\partial_r P) g_{rr} = 0$

Now, $g_{rr} = e^{2\nu}$

Recall, from constraint eqn we must have

$g_{rr} \cdot g_{tt} = -1$

$\therefore -e^{2\Phi} \cdot e^{2\nu} = -1$

$\therefore e^{2\nu} = e^{-2\Phi}$ ... (ii)

Now,

$u^t = e^{-\Phi}$

$\therefore$

$\Rightarrow (\rho + P)\Gamma^{r}_{tt} e^{-2\Phi + 2\nu} + (\partial_r P) e^{-2\nu} = 0$

From eq (ii)

Since $P \equiv P(r)$

$\therefore \partial_r P = \frac{dP}{dr}$

$\therefore \frac{dP}{dr} = -(\rho + P)\Gamma^{r}_{tt} e^{-2(\Phi - \nu)}$

$\Gamma^{r}_{\alpha\beta} = \frac{1}{2} g^{r\sigma}(g_{\beta\sigma,\alpha} + g_{\alpha\sigma,\beta} - g_{\alpha\beta,r})$

$= \frac{1}{2} g^{rr}(g_{\beta r,\alpha} + g_{\alpha r,\beta} - g_{\alpha\beta,r})$

$\therefore \Gamma^{r}_{tt} = \frac{1}{2} g^{rr}(0 + 0 - g_{tt,r})$

$= -\frac{1}{2} e^{-2\nu} \times -e^{2\Phi} \cdot 2 \cdot \Phi'$

$= e^{2(\Phi - \nu)} \frac{d\Phi}{dr}$

$\therefore \boxed{\frac{dP}{dr} = -(\rho + P)\frac{d\Phi}{dr}}$ □

(2) Linearised Metric

$g_{\mu\nu} = \eta_{\mu\nu} + \epsilon h_{\mu\nu}$

Show:

$g^{\mu\nu} = \eta^{\mu\nu} - \epsilon h^{\mu\nu}$

Ans Given

$g_{\alpha\beta} = \eta_{\alpha\beta} + \epsilon h_{\alpha\beta} + \mathcal{O}(\epsilon^2)$

Now

$g^{\mu\alpha} g_{\alpha\beta} = g^{\mu\alpha}\eta_{\alpha\beta} + \epsilon g^{\mu\alpha} h_{\alpha\beta} + \mathcal{O}(\epsilon)$

$\delta^{\mu}_{\beta} = g^{\mu\alpha}(\eta_{\alpha\beta} + \epsilon h_{\alpha\beta} + \mathcal{O}(\epsilon^2)) \times \eta^{\sigma\beta}$

$\rightarrow \eta^{\sigma\beta}\delta^{\mu}_{\beta} = g^{\mu\alpha}(\delta^{\sigma}_{\alpha} + \epsilon \eta^{\sigma\beta} h_{\alpha\beta} + \mathcal{O}(\epsilon^2))$

$\rightarrow \eta^{\sigma\mu} = g^{\mu\sigma} + g^{\mu\alpha}(\epsilon \eta^{\sigma\beta} h_{\alpha\beta} + \mathcal{O}(\epsilon^2))$

$\therefore$

$g^{\sigma\mu} = \eta^{\sigma\mu} - \epsilon \eta^{\sigma\beta} g^{\mu\alpha} h_{\alpha\beta} + \mathcal{O}(\epsilon^2)$ ... (i)

(*) $= \eta^{\sigma\mu} + \mathcal{O}(\epsilon)$.

So, $g^{\sigma\mu} \approx \eta^{\sigma\mu} - \epsilon \eta^{\sigma\beta} h_{\alpha\beta}(\eta^{\alpha\mu} + \mathcal{O}(\epsilon))$

$\therefore g^{\sigma\mu} = \eta^{\sigma\mu} - \epsilon \eta^{\alpha\mu}\eta^{\sigma\beta} h_{\alpha\beta} + \mathcal{O}(\epsilon^2)$

(*) ↑