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Q1) Show $\delta g = g\, g^{\mu\nu}\, \delta g_{\mu\nu} = -g\, g_{\mu\nu}\, \delta g^{\mu\nu}$

Ans We make use of the following identity:

for some $M \in M_n(\mathbb{R})$, and a parameter 't'

$$\frac{d}{dt}(\det M) = \det(M)\cdot \mathrm{Tr}\left(M^{-1}\frac{d}{dt}M\right)$$

Replace:

$d/dt \to \delta$

$M \to g_{\mu\nu}$

$\delta \det M \to g$.

$$\therefore \delta g = g \times \mathrm{Tr}\left[g^{\alpha\beta}\,\delta g_{\mu\nu}\right]$$

$$= g\left(g^{\mu\nu}\,\delta g_{\mu\nu}\right)$$ □

Now, $g^{\mu\nu} g_{\mu\nu} = 4 = \mathrm{Tr}(\mathbb{I})$

$\delta(g^{\mu\nu} g_{\mu\nu}) = 0$

$\to g^{\mu\nu}\,\delta g_{\mu\nu} = -g_{\mu\nu}\,\delta g^{\mu\nu}$

$\therefore \delta g = -g\left(g_{\mu\nu}\,\delta g^{\mu\nu}\right)$ □

▷ Proof: $\frac{d}{dt}(\det M) = \det(M)\cdot \mathrm{Tr}\left(M^{-1}\frac{dM}{dt}\right)$

$$\frac{d}{dt}\det(M) = \sum_i \begin{vmatrix} M_{11} & \cdots & \frac{d}{dt}M_{1i} & \cdots & M_{1n} \\ \vdots & & \vdots & & \vdots \\ M_{n1} & & \frac{d}{dt}M_{ni} & & M_{nn} \end{vmatrix}$$

Since, M is an invertible Matrix, it's determinant $\neq 0$

⇒ All the 'n' columns are linearly independent

⇒ forms a basis of $M_n(\mathbb{R})$.

if $M = [M_1 \cdots M_n]$

Then

$\frac{d}{dt}(M_i) = \sum_j C_{ij} M_j$

$\forall\, i \in \{1, 2, \cdots n\}$.

∴

RHS

$$= \sum_i \det\left[M_1 \cdots \sum_j C_{ij} M_j \cdots M_n\right]$$ (j in)

$$= \sum_i \sum_j C_{ij} \det[M_1 \cdots M_j \cdots M_n]$$

$\neq 0$ iff $i = j$

~~ie linearly $\neq 0$~~ o/w they are not linearly independent.

$$= \sum_i \sum_j C_{ij}\, \delta_{ij} \times \det(M_j)$$

$$= \sum_i C_{ii} \det M$$

$$= \mathrm{Tr}(C)\det M.$$

Now, look carefully

$\frac{d}{dt}(M_{ji}) = \sum_k C_{ik} M_{jk}$.

$\to \frac{d}{dt}[M] = C\,M$

$$\left[M^{-1}\frac{d}{dt}[M] = C\right]$$

∴ Hence Proved.

Q) Show

$$\delta(\sqrt{-g}) = -\frac{1}{2}(\sqrt{-g})\, g_{\mu\nu}\, \delta g^{\mu\nu}$$

using chain Rule, it's easy to see.

$$\delta(\sqrt{-g}) = +\frac{1}{2}\frac{\sqrt{-g}}{g} \cdot \delta g$$

$$= \frac{1}{2}\frac{\sqrt{-g}}{g}(-g)\, \delta g^{\mu\nu}\, g_{\mu\nu}$$

$$= -\frac{1}{2}\sqrt{-g}\; g_{\mu\nu}\, \delta g^{\mu\nu}$$

□

Q3) Show

$$g^{\mu\nu}\, \delta R_{\mu\nu} = \nabla_\mu A^\mu \quad \text{(Divergence)}$$

where

$$A^\mu = g^{\alpha\beta}\, \delta\Gamma^\mu_{\alpha\beta} - g^{\mu\nu}\, \delta\Gamma^\rho_{\nu\rho}$$

Ans $\delta R_{\mu\nu} = \delta R^\rho{}_{\lambda\rho\nu} = \delta\left(\partial_\rho \Gamma^\rho_{\nu\lambda} + \Gamma^\rho_{\rho\sigma}\Gamma^\sigma_{\nu\lambda} - \partial_\nu \Gamma^\rho_{\rho\lambda} - \Gamma^\rho_{\nu\sigma}\Gamma^\sigma_{\rho\lambda}\right)$

$$= \partial_\rho \delta\Gamma^\rho_{\nu\lambda} + \delta(\Gamma^\rho_{\rho\sigma}\Gamma^\sigma_{\nu\lambda}) - \partial_\nu \delta\Gamma^\rho_{\rho\lambda} - \delta(\Gamma^\rho_{\nu\sigma}\Gamma^\sigma_{\rho\lambda})$$

$$= \partial_\rho \delta\Gamma^\rho_{\nu\lambda} + \delta\Gamma^\rho_{\rho\sigma}\Gamma^\sigma_{\nu\lambda} + \delta\Gamma^\sigma_{\nu\lambda}\Gamma^\rho_{\rho\sigma} - \partial_\nu \delta\Gamma^\rho_{\rho\lambda} - \delta\Gamma^\rho_{\nu\sigma}\Gamma^\sigma_{\rho\lambda} - \delta\Gamma^\sigma_{\rho\lambda}\Gamma^\rho_{\nu\sigma}$$

Now consider covariant derivative of variation of the Christoffel Symbol —

$$\nabla_\rho(\delta\Gamma^\rho_{\nu\lambda}) = \partial_\rho(\delta\Gamma^\rho_{\nu\lambda}) + \underbrace{\Gamma^\rho_{\rho\sigma}\delta\Gamma^\sigma_{\nu\lambda}}_{\text{For } \rho} - \underbrace{\Gamma^\sigma_{\rho\nu}\delta\Gamma^\rho_{\sigma\lambda}}_{\text{For } \nu \text{ index}} - \underbrace{\Gamma^\sigma_{\rho\lambda}\delta\Gamma^\rho_{\nu\sigma}}_{\text{For } \lambda}$$

- $\nabla_\nu(\delta\Gamma^\rho_{\sigma\lambda}) = \partial_\nu(\delta\Gamma^\rho_{\sigma\lambda}) + \Gamma^\rho_{\nu\tau}\delta\Gamma^\tau_{\sigma\lambda} - \Gamma^\tau_{\nu\sigma}\delta\Gamma^\rho_{\tau\lambda} - \Gamma^\tau_{\nu\lambda}\delta\Gamma^\rho_{\sigma\tau}$

observe that,

$$\delta R_{\mu\nu} = \nabla_\rho(\delta\Gamma^\rho_{\nu\lambda}) - \nabla_\nu(\delta\Gamma^\rho_{\rho\lambda})$$

Recall $\nabla_\alpha g^{\mu\nu} = 0$.

$\rightarrow$

$$\therefore g^{\lambda\nu}\delta R_{\lambda\mu} = \nabla_\rho(g^{\lambda\nu}\delta\Gamma^\rho_{\nu\lambda}) - \nabla_\nu(g^{\lambda\nu}\delta\Gamma^\rho_{\rho\lambda})$$

$$= \nabla_\mu(g^{\lambda\nu}\delta\Gamma^\mu_{\nu\lambda}) - \nabla_\mu(g^{\lambda\mu}\delta\Gamma^\rho_{\rho\lambda})$$

From this we obtain.

$$g^{\lambda\nu}\delta R_{\lambda\nu} = \nabla_\mu(g^{\lambda\nu}\delta\Gamma^\mu_{\nu\lambda} - g^{\lambda\mu}\delta\Gamma^\rho_{\rho\lambda})$$

$$= \nabla_\mu A^\mu$$

□