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Assignment - 2

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**Q1)** Show that the quantity $g_{\alpha\beta} V^\alpha V^\beta$ is a scalar, if $g$ is a metric tensor & $\vec{V}$ is a contravector.

**Ans** By definition, if a mathematical object does NOT transform under a coordinate transformation, then it is called a tensor of rank (0,0) or a SCALAR.

~~Now,~~ ⇒ We have T.P $\|\vec{V'}\|^2 = \|\vec{V}\|^2$.

$\|\vec{V'}\|^2 = \vec{V'} \cdot \vec{V'} = g'_{\alpha\beta} V'^\alpha V'^\beta$

Contravector ⇒ Transforms opp to that of a vector.

Let $Y$ = denote primed coordinate
$X$ = " unprimed "

$\therefore (V')^\alpha = \frac{\partial Y^\alpha}{\partial X^\mu} V^\mu$ | $(V')^\beta = \frac{\partial Y^\beta}{\partial X^\nu} V^\nu$

$g'_{\alpha\beta} = \frac{\partial X^\mu}{\partial Y^\alpha} \frac{\partial X^\nu}{\partial Y^\beta} g_{\mu\nu}$

Subst. $= \left(\frac{\partial X^\mu}{\partial Y^\alpha} \frac{\partial X^\nu}{\partial Y^\beta} g_{\mu\nu}\right)\left(\frac{\partial Y^\alpha}{\partial X^\mu} V^\mu\right)\left(\frac{\partial Y^\beta}{\partial X^\nu} V^\nu\right)$

$= 1 \cdot 1 \cdot g_{\mu\nu} V^\mu V^\nu$

$= V^\mu V^\nu (e_\mu \cdot e_\nu)$

$= (V^\mu e_\mu) \cdot (V^\nu e_\nu)$ → coordinate basis vectors (Defn of metric)

$= \vec{V} \cdot \vec{V} = \|\vec{V}\|^2$

□

Q3) Show that the dot product

$$\vec{A}\cdot\vec{B} = g_{\mu\nu} A^\mu B^\nu \quad \text{is scalar.}$$

Ans Exactly same as in Q1)

$$\vec{A'}\cdot\vec{B'} = g'_{\alpha\beta} (A')^\alpha (B')^\beta$$

$$= \left(\frac{\partial x^\mu}{\partial y^\alpha}\cdot\frac{\partial x^\nu}{\partial y^\beta} g_{\mu\nu}\right)\left(\frac{\partial y^\alpha}{\partial x^\mu} A^\mu\right)\left(\frac{\partial y^\beta}{\partial x^\nu} B^\nu\right)$$

$$= 1 \cdot 1 \cdot g_{\mu\nu} \cdot A^\mu B^\nu$$

$$= \vec{A}\cdot\vec{B} \qquad \text{D}$$

This is Not transformed under a coordinate transformation, ∴ It is a scalar.

Q.2) Show

$$(g^{-1'})^{\mu\nu} = \Lambda^\mu_{\ \alpha}\,\Lambda^\nu_{\ \beta}\,(g^{-1})^{\alpha\beta}$$

Ans Since we would like to reserve $\Lambda$ for Lorentz transformations only, let us denote general transformations with $L$.

s.t

$$L^\mu_{\ \alpha} = \frac{\partial x'^\mu}{\partial x^\alpha} \;;\; L_\mu^{\ \alpha} = \frac{\partial x^\alpha}{\partial x'^\mu} = (L^{-1})^{\mu'}_{\ \alpha}.$$

where, we define the metric inverse $g^{\rho\sigma}$ of $g_{\rho\sigma}$ as

$$\boxed{g^{\rho\sigma} g_{\sigma\nu} = \delta^\rho_{\ \nu}}$$

So, we have to show in our new notation.

T.P,

$$\boxed{g'^{\mu\nu} = L^\mu_{\ \alpha} L^\nu_{\ \beta} g^{\alpha\beta}}$$

↳ metric in new coordinates $(x' (x^1, x^2, \ldots))$

To begin with note,

Assume: $X = X(x^1, x^2, \ldots x^n)$

coordinate bases are linearly independent

s.t.

$$\therefore \boxed{L^{\mu'}{}_{\nu} \, L_{\mu'}{}^{\rho} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} \cdot \frac{\partial x^{\rho}}{\partial x'^{\mu}} = \frac{\partial x^{\rho}}{\partial x^{\nu}} = \delta^{\rho}_{\nu}}$$

Using this, note that.

**Claim:** $\delta^{\rho}_{\sigma}$ is an invariant quantity in any transformation!

$$\delta'^{\rho}_{\sigma} = L^{\rho}{}_{\nu} \, L_{\sigma}{}^{\mu} \, \delta^{\nu}_{\mu}$$

$$= L^{\rho}{}_{\nu} \, L_{\sigma}{}^{\nu} \quad \ldots (*)$$

$$= \delta^{\rho}_{\sigma}$$

□

Now, in any frame, we can write,

(comes from invariancy of $ds^2 = g_{\mu\nu} dx^{\mu} dx^{\nu}$)

$$g^{\rho\sigma} g_{\rho\nu} = (g')^{\rho\sigma} (g')_{\rho\nu} = \delta'^{\sigma}{}_{\nu} = \delta^{\sigma}{}_{\nu}$$

(∵ proved above)

And,

Recall, $(g')_{\rho\nu} = L_{\rho}{}^{\alpha} L_{\nu}{}^{\beta} g_{\alpha\beta}$.

$\therefore$

$$(g')^{\rho\sigma} L_{\rho}{}^{\alpha} L_{\nu}{}^{\beta} g_{\alpha\beta} = \delta^{\sigma}_{\nu}$$

$\times L^{\nu}{}_{k}$ both sides. (✓ as $\nu \to$ free).

$$\Rightarrow (g')^{\rho\sigma} L_{\rho}{}^{\alpha} (L_{\nu}{}^{\beta} L^{\nu}{}_{k}) g_{\alpha\beta} = \delta^{\sigma}_{\nu} L^{\nu}{}_{k}$$

$$= L^{\sigma}{}_{k}$$

$$\to (g')^{\rho\sigma} L_{\rho}{}^{\alpha} \delta^{\beta}_{k} g_{\alpha\beta} = L^{\sigma}{}_{k}$$

$$\to (g')^{\rho\sigma} L_{\rho}{}^{\alpha} g_{\alpha k} = L^{\sigma}{}_{k}$$

$\times$ g$^{-1}$ both sides.

$$(g')^{\rho\sigma} L_\rho{}^\alpha (g_{\alpha\kappa} g^{\kappa r}) = L^\sigma{}_\kappa g^{\kappa r}$$

$$\rightarrow (g')^{\rho\sigma} L_\rho{}^\alpha \delta_\alpha{}^r = L^\sigma{}_\kappa g^{\kappa r}.$$

$$\Rightarrow (g')^{\rho\sigma} L_\rho{}^r = L^\sigma{}_\kappa g^{\kappa r}.$$

$\Rightarrow$ As $\rho$ is not free

$L^m{}_r$ both side.

$$(g')^{\rho\sigma} (L_\rho{}^r \cdot L^m{}_r) = L^\sigma{}_\kappa L^m{}_r \cdot g^{\kappa r}$$

$$(g')^{\rho\sigma} \delta_\rho{}^m = L^\sigma{}_\kappa L^m{}_r g^{\kappa m}$$

$$(g')^{m\sigma} = L^m{}_r L^\sigma{}_\kappa g^{r\kappa}$$ (Asymmetric symmetric)

$$\Rightarrow (g')^{\mu\nu} = L^\mu{}_\alpha L^\nu{}_\beta g^{\alpha\beta}.$$

(Rearranging indices) □