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PH4113 – Assignment (3)

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Q1) Show that $\Gamma^{\mu}_{\alpha\beta} = \Gamma^{\mu}_{\beta\alpha}$. i.e $\Gamma^{\mu}_{\alpha\beta}$ is symmetric ~~under~~ in lower indices.

Ans Ref [1],

Method-2

from definition.

$$\Gamma^{\mu}_{\alpha\beta} = L_{\rho}{}^{\mu}\left(L^{\rho}{}_{\alpha,\beta}\right)$$

$L_{\rho}{}^{\mu}$ is the coordinate transformation ~~metric~~ matrix wrt some metric $g_{\mu\nu}$.

$$\therefore \Gamma^{\mu}_{\alpha\beta} = L_{\rho}{}^{\mu} \cdot \frac{\partial}{\partial x^{\beta}}\left(L^{\rho}{}_{\alpha}\right)$$

=

By definition

$$\left| L^{\rho}{}_{\alpha} = \frac{\partial x'^{\rho}}{\partial x^{\alpha}} \right| =$$

=

$$= L_{\rho}{}^{\mu} \cdot \frac{\partial}{\partial x^{\beta}}\left(\frac{\partial x'^{\rho}}{\partial x^{\alpha}}\right)$$

$$= L_{\rho}{}^{\mu} \frac{\partial}{\partial x^{\alpha}}\left(\frac{\partial x'^{\rho}}{\partial x^{\beta}}\right) = L_{\rho}{}^{\mu}\, \partial_{\alpha} L^{\rho}{}_{\beta}$$

$$= L_{\rho}{}^{\mu} L^{\rho}{}_{\beta,\alpha}$$

$$= \Gamma^{\mu}_{\beta\alpha} \quad \square$$

Method 3

Note that, for any scalar ϕ.

$\bar{\nabla}\bar{\nabla}\phi \equiv \nabla_\alpha \nabla_\beta \phi \; \tilde{\omega}^\alpha \otimes \tilde{\omega}^\beta$.

↑ components of tensor,

≡ symmetric on exchange of α, β.

$\because$ $\bar{\nabla}\bar{\nabla}\phi$ is a tensorial quantity → remains same in all frames.

$\therefore$

$$\nabla_\alpha \nabla_\beta \phi = \nabla_\beta \nabla_\alpha \phi$$

(∇_α = covariant Derivative)

$$\nabla_\alpha (\partial_\beta \phi) = \nabla_\beta (\partial_\alpha \phi)$$

$$\Rightarrow \partial_\alpha \partial_\beta \phi - \Gamma^\mu_{\alpha\beta} \partial_\mu \phi = \partial_\beta \partial_\alpha \phi - \Gamma^\mu_{\beta\alpha} \partial_\mu \phi$$

$$\Rightarrow (\Gamma^\mu_{\beta\alpha} - \Gamma^\mu_{\alpha\beta}) \partial_\mu \phi = 0$$

For it to hold for any arbitrary ϕ, we must have $\Gamma^\mu_{\beta\alpha} = \Gamma^\mu_{\alpha\beta}$.