Solutions attempted by
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HW Problem Set

Note: Meaning of a variable is same as we use in the class.

1. Discuss how astronomers use Kepler's laws to determine individual star mass of a binary system when the plane of orbit has an inclination angle i w.r.t. plane of sky. During class, we discussed the case when $i = 0$.

2. Consider two stars in orbit about a mutual center of mass. If a_1 is the semajor axis of the orbit of star of mass m_1 and a_2 is the semimajor axis of the orbit of star of mass m_2, prove that the semimajor axis of the orbit of the reduced mass is given by $a = a_1 + a_2$.

3. Do problem 7.4 of Carroll and Ostlie.

4. Two stars, A and B, have following epoch J2000.0 RA (α) and DEC (δ):
A - $\alpha = 14^h29^m42.95^s, \delta = -62°40'46.1''$
B - $\alpha = 14^h39^m36.50^s, \delta = -60°50'02.3''$

a) Find out the angular separation between A and B.
b) If the distance to A is 4×10^{16}m, then how far is A from B?

5. Sun's apparent magnitude is -26.83. What will be its absolute magnitude? A star at a distance 4 pc has an apparent magnitude 2. What will be its absolute magnitude? Given that the Sun has a luminosity 3.9×10^{26} W, find the luminosity of the distant star using above information.

6. Communication and weather satellites are often placed in geosynchronous orbits above Earth. These are orbits where satellites can remain fixed above a specific point on the Earth surface. At what altitude must these satellites be located?

7. We showed that specific intensity remains constant along a ray path. Show that net flux $F_\nu = \int I_\nu d\Omega$ falls of as $1/r^2$.

8. In the class, we derived the hydrostatic equilibrium by considering gravity and pressure gradient. Consider a star is also rotating with a constant angular velocity Ω about a rotation axis. Is it possible to develop the equation of hydrostatic equilibrium for this configuration?

9. Assuming the Sun has radiated away half of its gravitational energy content, show that it's lifetime comes out to be around 10^7 years. You can find the requireed data (Sun mass, radius and radiative energy loss rate) by Google search!

10. We used the following formula during the derivation of dT/dr for radiation dominated part of stellar structure:

$$\frac{dP_{rad}}{dr} = -\frac{\kappa\rho}{c}F_{rad}$$

Derive this. Please see Carroll and Ostlie book section 9.4 and equation 9.42 for this.

11. Assuming the opacity in very hot stars is provided by Thomson scattering, show that L/M has to be less than a critical value and find its numerical value. (hint: use Eddington limit formula). How does this value compare for the Sun?

12. During the class, we mentioned that, to overcome the Coulomb barrier for fusion of two protons inside a plasma, required temperature $\mathrm{T}_{classical}$ is 10^{10} K. Prove this mathematically. Look up around eq. 10.26 of Carroll and Ostlie book.

Argue that corresponding temprature for quantum mechanical tunneling is 10^7 K.

13. The Q value of a reaction is the amount of energy released (or absorbed) during the reaction. Calculate the Q value for each step of PP1 chain. Express the results in MeV. Masses of ${}^1_1$H, ${}^2_1$H, ${}^3_2$He and ${}^4_2$He are 1.0078, 2.0141 amu, 3.0160 amu and 4.0026 amu respsectively. Neglect positron/neutrino masses.

What fraction of hydrogen mass got converted into liberated energy?

14. One can make rough estimate of main sequence lifetime T_{ms} of a star using above result of Prob 13. Assuming the Sun will move off the main sequence when central 10% hydrogen will get converted into helium, estimate it's T_{ms}.
Now using observed mass-luminosity relationship $L \propto M^x$ with $x \approx 3.5$ from H-R diagram and same assumption as above, make estimate of T_{ms} for lower end and upper end star of main sequence (approx 0.1 $M_\odot$ and 85 $M_\odot$).

15. Standard solar model provides the core density ρ_c and temperature T_c as $\rho_c = 1.53 \times 10^5$ kg/m^3 and $T_c = 1.57 \times 10^7$ K. Show that electron degenracy condition is not satisfied for the Sun.

For Sirius B, $\rho_c = 3.0 \times 10^9$ kg/m^3 and $T_c = 7.6 \times 10^7$ K are estimated. Show that electron degenracy is achieved for Sirius B.

16. Pulsars are visualized as fast rotating neutron stars with strong magnetic dipole. And such magnetic dipole radiates energy! It is believed that this energy can somehow be converted to kinetic and radiative energy of the surrounding nebula. And because of this energy dissipation, the neutron star also slows down at a certain observable rate.

Observationally, it is found that The Crab Nebula emits about 10^{31} W and it is believed that above mechnism is responsible for that. Assume the neutron star at the center of Crab Nebula to be a rigid sphere, mass 1 $M_\odot$, radius 10 KM. It is observationally found that the pulsar has pulse time-period of 0.03 sec. Using all these information, calculate what should be the slowdown rate in pulse time-period?

Compare this with observed rate of slowdown for Crab pulsar $\approx 10^{-13}$ s/s.

17. In liquid-drop model of an atomic nucleus, a nucleus with mass number A has a radius of $r_0 A^{1/3}$, where $r_0 = 1.2 \times 10^{-15}$ m. Find the density of this nuclear model. Compare this with density of different layers of a neutron star.

18. Consider two point masses, each having mass m, that are separated vertically by a distance of 1 cm just above the surface of a neutron star of radius R and mass M. Using Newton's law of gravity, find an expression for the ratio of the gravitational force on the lower mass to that on the upper mass, and evaluate this expression for R = 10 km, M = 1.4 $M_\odot$, and m = 1 g.

Compare this in case of the Earth.

19. In the class, we calculated collisional relaxation time for self-gravitating

system. Follow ARC book and express this time in the following form (Eq. 7.12 of ARC book):

$$T_{rel} = 10^{10}\frac{v^3}{n}$$

assuming $m = M_\odot$. Then calculate T_{rel} values given in Table 7.1.

20. We derived Euler's equation of fluid dynamics in class. Starting from that, derive Bernoulli's principle for steady flow:

$$\frac{1}{2}v^2 + \int \frac{dp}{\rho} + \Phi = \text{constant},$$

where, Φ is gravitational potential.

There are fluid whose density ρ may not change appreciably during their evolution (e.g., water or air). Such fluids are called incompressible flow (can not be compressed!). Show that for such flow, one can write

$$\frac{1}{2}v^2 + \frac{p}{\rho} + \Phi = \text{constant},$$

21. Show that the mass continuity and the Euler's equations can be combined to produce the following momentum density conservation form for momentum density component ρv_i ($i \equiv x, y, z$ components):

$$\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j}(\rho v_i v_j + P\delta_{ij}) = \rho F_i$$

22. We derived following induction equation during the class:

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times (\vec{v} \times \vec{B}) + \eta \nabla^2 \vec{B},$$

with $\eta = \frac{1}{\mu_0 \sigma}$. Show that if we consider ideal condition, i.e., $\eta = 0$, we can derive

$$\frac{d}{dt}\int_S \vec{B}.d\vec{S} = 0$$

This tells about the magnetic flux conservation accross any arbitrary surface element S as S moves inside fluid. However, if η is non-zero (i.e., resistive MHD), there is no such conservation.

HW Problem set

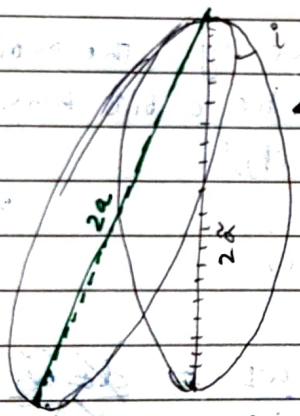
① Discuss how astronomers use Kepler's Laws to determine individual star mass of a binary system when the plane of orbit has an inclination angle i wrt plane of the sky.

Ans Recall from Kepler's Law

$$T^2 = \frac{4\pi^2}{G(m_1+m_2)} a^3$$

where $a = a_1 + a_2$.

a_i = semi Major axis of i'th planet's orbit.



Project orbit that we observe on the sky

But we donot actually "observe" the distance, instead what we see are the projections of their angular separation.

Let "d" the distance to the binary system be known by parallax.

α_i = Denote the Actual Angular seperation of the inclined orbit.

$\tilde{\alpha}_i$ = Denotes the Angular separation in the projected orbit.

~~Clearly~~ :

$$a_1 + a_2 = a$$

~~$da_1 \cos i + da_2 \cos i = d$~~

$= \tilde{a}_1$ $= \tilde{a}_2$

$$\Rightarrow a_1 \cos i + a_2 \cos i = a \cos i = \tilde{a}$$

$$\Rightarrow (\alpha_1 d)\cos i + \alpha_2 d \cos i = a \alpha \cos i$$

$$\Rightarrow \tilde{\alpha}_1 + \tilde{\alpha}_2 = \tilde{\alpha}$$

∴ we substitute.

$$a = d\alpha = \left(\frac{d}{\cos i}\right)\tilde{\alpha}$$

Now, come to COM frame of the orbit.
(We assume both the orbits have same eccentricity)

$$m_1 \bar{r}_1 + m_2 \bar{r}_2 = 0$$

$$r = \frac{a(1-e^2)}{1 + e\cos\theta}$$

$$\therefore \frac{m_1}{m_2} = \frac{r_2}{r_1} = \frac{a_2}{a_1} = \frac{d\tilde{\alpha}_2}{d\tilde{\alpha}_1}$$

Magnitude wise

∴ Our observable = $\tilde{\alpha}_1, \tilde{\alpha}_2$; i, T, d

∴ The equation becomes.

$$T^2 = \frac{4\pi^2}{G(m_1+m_2)} \left(\frac{d}{\cos i}\right)^3 \tilde{\alpha}^3$$

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$$m_1 + m_2 = \frac{4\pi^2}{GT^2}\left(\frac{a}{\cos i}\right)^3 \tilde{\alpha}^3.$$

Substitute.

$$m_1 = m_2\left(\frac{\tilde{\alpha}_2}{\tilde{\alpha}_1}\right)$$

$$\Rightarrow \quad m_2 = \frac{4\pi^2}{GT^2}\left(\frac{a}{\cos i}\right)^3 \frac{(\tilde{\alpha}_1 + \tilde{\alpha}_2)^2}{\tilde{\alpha}_1}$$

$$\therefore \quad \boxed{m_2 = \frac{4\pi^2}{GT^2}\left(\frac{a}{\cos i}\right)^3 \tilde{\alpha}_1 (\tilde{\alpha}_1 + \tilde{\alpha}_2)^2}$$

Similarly $$\boxed{m_1 = \frac{4\pi^2}{GT^2}\left(\frac{a}{\cos i}\right)^3 \tilde{\alpha}_2 (\tilde{\alpha}_1 + \tilde{\alpha}_2)^2}$$

Q4). Two stars A, and B have following epoch J2000 RA(α) and DEC(δ).

(A) $\alpha_A = 14^h\ 29^m\ 42.95^s$, $\delta_A = -62^\circ\ 40'\ 46.1''$

$= (14\times 15) + (29\times 0.25) + \left(42.95\times \frac{0.25}{60}\right)$

$= 217.4289^\circ$ $\quad = -62.6795^\circ$

(B) $\alpha_B = 14^h\ 39^m\ 36.50^s$, $\delta_B = -60^\circ\ 50'\ 2.3''$

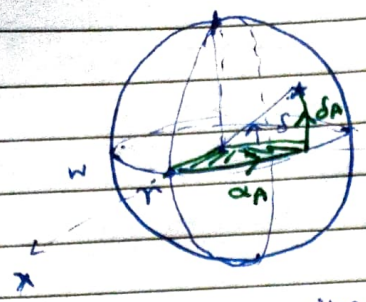
$= (14\times 15) + (39\times 0.25) + (36.5\times 0.25/60)$ $\quad = -60.8339^\circ$

$= 219.9021^\circ$

Conversions		
$24^h = 360^\circ$	$60^m = 15^\circ$	$60^s = 0.25^\circ$
$1^h = 15^\circ$	$1^m = 0.25^\circ$	$1^s = 4.1667 \times 10^{-3}$ deg

Right Ascension ~~increase~~ increases eastward.

$\gamma \rightarrow 0^h\ 0^m\ 0^s$; Dec $= 0^\circ$ (as at celestial Equator)



We can write the coordinate unit vectors of the star's position on the celestial sphere as.

$$\hat{r}_A = \begin{pmatrix} \cos\delta_A \cos\alpha_A \\ \cos\delta_A \sin\alpha_A \\ \sin\delta_A \end{pmatrix}$$

Similarly $\hat{r}_B$ follows.

Let the angular separation between these 2 vectors be $\Delta\varphi$.

Then. $\hat{r}_A \cdot \hat{r}_B = \cos\Delta\varphi$.

⇒) In Eudidean Metric,

$$\cos\Delta\varphi = \cos\delta_A \cos\alpha_A \cos\delta_B \cos\alpha_B + \cos\delta_A \sin\alpha_A \cos\delta_B \sin\alpha_B + \sin\delta_A \sin\delta_B$$

$$= \cos\delta_A \cos\delta_B [\cos(\alpha_A - \alpha_B)] + \sin\delta_A \sin\delta_B.$$

⊙ Substituting values we get

$$= 0.99927\ldots$$

Angular separation $\Delta\varphi = \cos^{-1}(\quad) =$

Note if individual distances are known.

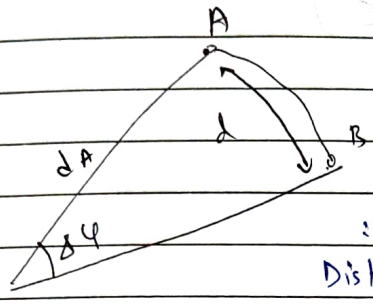
dist. b/w stars can be written as

$$d^2 = d_1^2 + d_2^2 + 2 d_1 d_2 \cos\theta$$

Angular separation between them is -

2.185055 deg · or θ = 0.381364 rad

(b) If distance to A' is 4×10^{16} m, how far is A from B.



Given

$d_A = 4 \times 10^{16}$ m.

∴ Dist of A from B is

$\triangleq d_A \cdot \Delta\varphi$ rad

$= \frac{4 \times 10^{16} \times 0.381364}{9.5 \times 10^{15}}$

$= \frac{4 \times 3.81364}{9.461}$

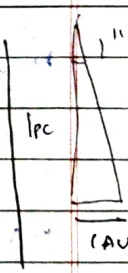
$= 1.612$ ly

1 ly $\triangleq 9.5 \times 10^{15}$ m

(5) Given,

(a) $m_\odot = -26.83$, $d = 1$ AU $= 4.8481 \times 10^{-6}$ pc

$M_\odot = ?$



(b) Apparent Magnitude

$m_\odot = M_\odot + 5 \log_{10}\left(\frac{d}{10\,pc}\right)$

$\therefore M_\odot = m_\odot - 5 \log_{10}\left(\frac{d}{10\,pc}\right)$

$= -26.83 - 5 \log_{10}(4.8481 \times 10^{-7})$.

$2\pi \to 360°$

$1 \to \frac{360}{2\pi}$

$1 \to \frac{\pi}{180}$ rad

$= 4.1742$ mag

1 AU = 1 pc × 1"

$= 1 \times \left(\frac{1}{3600} \times \frac{\pi}{180}\right)$

$= 4.8481 \times 10^{-6}$ pc

(b) $m_* = 2$

$d_* = 4\,pc$

$M_* = ?$

$$M_* = m_* - 5\log_{10}\left(\frac{d_*}{10\,pc}\right)$$

$$= 2 - 5\log_{10}\left(\frac{4\,pc}{10\,pc}\right)$$

$$= 3.09897 \text{ mag}$$

(c) Given $L_\odot = 3.9 \times 10^{26}$ W.

~~we know~~ By definition, we define ~~absolute~~ apparent magnitude as:

$$m_1 = -2.5\log_{10} f_1 + C_x$$

$$m_2 = -2.5\log_{10} f_2 + C_x$$

C_x → some constant dependent on the wave band

$$\Rightarrow m_1 - m_2 = -2.5\log_{10}\left(\frac{f_1}{f_2}\right)$$

Now, $$f_1 = \frac{L_1}{4\pi d_1^2}$$

$$m_1 - m_2 = -2.5\log_{10}\left(\frac{L_1}{L_2}\frac{d_2^2}{d_1^2}\right)$$

$$= -2.5\log_{10}\left(\frac{L_1}{L_2}\right) + 5\log_{10}\left(\frac{d_1}{d_2}\right)$$

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$$\Rightarrow m_\odot - m_\star = -2.5 \log_{10}\left(\frac{L_\odot}{L_\star}\right) + 5\log_{10}\left(\frac{d_\odot}{d_\star}\right)$$

$$\Rightarrow -2.5 \log_{10}\left(\frac{L_\odot}{L_\star}\right) = -26.83 - 2 - 5\log_{10}\left(\frac{1.212}{4.8487 \times 10^{-6}}\right)$$

$$= 0.752486$$

$$\Rightarrow \frac{L_\odot}{L_\star} \simeq 10^{-0.301}$$

$$L_\star = L_\odot 10^{0.301}$$

$$= 1.99986\, L_\odot$$

$$\simeq 2\, L_\odot$$

$$= 7.8 \times 10^{26}\ W$$

(6) Communication and weather satellites are often placed in Geosynchronous orbits above Earth. These are orbits where satellite can remain fixed above a specific point on Earth surface. At what altitude must these satellites be located?

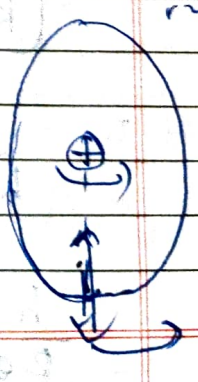
Let Altitude = h (Km)

$R_\oplus = 6400$ Km

$M_\oplus = 6 \times 10^{24}$ kg

$G = 6.67 \times 10^{-11}\ \frac{N - m^2}{kg^2}$

$F = \frac{GM^2}{r^2}$



Basically

$\omega_\oplus = \omega_{satellite}$.

i.e Angular velocities should match.

However, one more condition, must also be satisfied, for a stable orbit around earth.

$$\frac{GMm}{(R+h)^2} = m\omega^2 (R+h)$$

$$= m\omega^2 \cdot (R+h)^3$$

Put

$\omega = \omega_\oplus$

$24^h \rightarrow 360^\circ$

$1s \rightarrow \frac{360}{24 \times 3600} \times \frac{\pi}{180}$ rad.

$\omega_\oplus = 7.2722 \times 10^{-5}$ rad/s.

$$\frac{GM_\oplus}{\omega_\oplus^2} = (R_\oplus + h)^3$$

Since, $h > 0$.

we should have.

$$(R_\oplus + h) = \sqrt[3]{\frac{GM_\oplus}{\omega_\oplus^2}}$$

$$h = \sqrt[3]{\frac{GM_\oplus}{\omega_\oplus^2}} - R_\oplus$$

$$\frac{h}{R_\oplus} = \sqrt[3]{\frac{GM_\oplus}{R_\oplus^3 \omega_\oplus^2}} - 1$$

$= g$

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$$(R_\oplus + h)^3 = + \left(\frac{G M_\oplus}{\omega_\oplus^2}\right)^{1/3}$$

$$\therefore h = \left(\frac{G M_\oplus}{\omega^2_\oplus}\right)^{1/3} - R_\oplus$$

$$\frac{h}{R_\oplus} = \left(\frac{G M_\oplus}{R_\oplus^3 \omega^2_\oplus}\right)^{1/3} - 1.$$

Since, $g = \frac{G M_\oplus}{R_\oplus^2} = 9.8$

$$T = \frac{2\pi}{\omega}$$

$$\frac{m v^2}{r} = \frac{G M_\oplus m}{r^2}$$

$$v = \frac{2\pi r}{T}.$$

$$\left(\frac{2\pi}{T}\right)^2 r^2 = \frac{G M_\oplus}{r}$$

$$r_\oplus = \sqrt[3]{\frac{G M_\oplus}{4\pi^2} T^2}$$

$$r_\oplus = \sqrt[3]{\frac{6.67 \times 10^{-11} \times 6 \times 10^{24} \times 24^2 \times 3600^2}{4 \times \pi^2}}$$

$$= 42311825.27 \text{ m}$$

$$\frac{h}{R_\oplus} = \frac{}{}$$

$$1^h \rightarrow 360^\circ$$

$$R_\oplus + h = 42311.825 \text{ km}$$

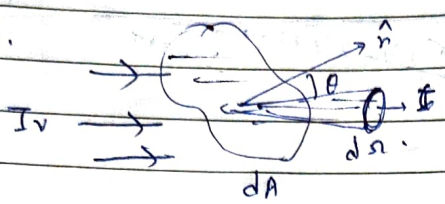
Alt:- $h = \frac{39911.825}{6.4 \times 10^3} \approx 6 R_\oplus$

7.Q) In a static, absorbtion free universe, we showed that specific Intensity remains constant along a ray path.

$\int I_\nu \cos\theta \, d\Omega$.

Show that net flux $F_\nu =$ ~~$\int I_\nu \, d\Omega$~~, falls off as $\frac{1}{r^2}$.

Ans.



Specific intensity is defined as

$$I_\nu = \frac{dE_\nu}{dt \, d\nu \, d\vec{\Omega} \cdot d\vec{A}}$$

(Total energy)

Also called Brightness

$$= \frac{dE_\nu}{d\nu \, dt \, dA \, d\Omega \cos\theta}$$

θ = Angle b/w ~~direction~~ normal to surface and direction of propagation.

Suppose now, that we have a radiation field (rays in all directions), ~~and construct a small area~~ and we have a small area element $d\vec{A}$ at some arbitrary orientation $\hat{n}$, Then the differential amount of flux from solid angle $d\Omega$ is

$$dF_\nu \left(\text{erg s}^{-1}\text{ cm}^{-2}\text{ Hz}^{-1}\right) = I_\nu \cos\theta \, d\Omega.$$

The Net Flux in direction $\hat{n}$, $F_\nu(\hat{n})$ is obtained by

$$F_\nu = \int I_\nu \cos\theta \, d\Omega.$$

$$= \int I_\nu(\theta, \phi) \cos\theta \, d\Omega.$$

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$I_\nu \equiv I_\nu(\vec{r}, t, \hat{n})$

If I_ν is isotropic,

i.e $I_\nu(\theta, \phi) = I_{\nu,iso}$ constant

Then, as $\int \cos\theta \, d\Omega = 0$

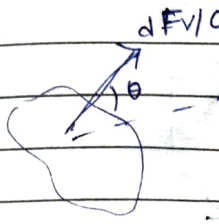
$F_{\nu,iso} = 0$.

Momentum Flux / Radiation pressure.

To get the flux of momentum normal to dA,

Momentum of photon $= E/c$

$\therefore$ " flux along ray at angle $\theta = \frac{dF_\nu}{c}$

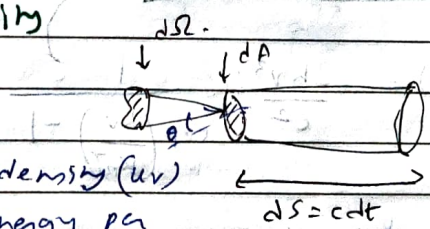


$\therefore$ Component of momentum flux normal to dA.

$$p_\nu \left(\text{dynes cm}^{-2}\ \text{Hz}^{-1}\right) = \frac{1}{c}\int I_\nu \cos^2\theta \, d\Omega$$

F_ν, p_ν are called Moments of I_ν.

Radiative Energy Density



~~To determine~~

The specific energy density (u_ν) is defined as the energy per unit volume per unit frequency range.

To determine this it is convenient to consider first the energy density per unit solid angle $u_\nu(\Omega)$ by

$dE = u_\nu(\Omega)\, dV\, d\Omega\, d\nu$ where $dV = dA \times c\,dt$ volume of cylinder.

$\therefore dE = u_\nu(\Omega)\, c\, dA\, dt\, d\Omega\, d\nu \times \cos\theta$

Also, I recall, $dE = I_\nu\, dA\, d\Omega \cos\theta\, dt\, d\nu$.

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Equating gives

$$u_\nu(\Omega) = \frac{I_\nu}{c}$$

∴ To get, to total energy density of radiation at a point associated with frequency ν, we integrate it from diff directions

∴

$$\boxed{U_\nu = \int \frac{I_\nu}{c} d\Omega}$$

For blackbody / Any Isotropic Radiation field.

$$I_\nu = B_\nu(T)$$

$$U_\nu = B_\nu(T) \int \frac{d\Omega}{c}$$

$$\boxed{U_\nu = \frac{4\pi}{c} B_\nu(T)}$$

Define quantity Mean Intensity

$$\boxed{J_\nu = \frac{1}{4\pi} \int I_\nu \, d\Omega}$$

$$\boxed{U_\nu = \frac{4\pi}{c} J_\nu}$$

$$B_\nu(T) = \frac{2h\nu^3}{c^2} \frac{1}{\exp\left(\frac{h\nu}{k_B T}\right) - 1}$$

Radiation Pressure

Now,

$$P_\nu = \frac{1}{c} \int I_\nu \cos^2\theta \, d\Omega.$$

isotropic field

$$= \frac{I_\nu}{c} \int \cos^2\theta \, d\Omega.$$

$$= \frac{I_\nu}{c} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \cos^2\theta \sin\theta \, d\theta \, d\phi$$

$$= 2\pi \left(\frac{I_\nu}{c}\right) \int_0^{\pi} \cos^2\theta \, (\sin\theta \, d\theta)$$

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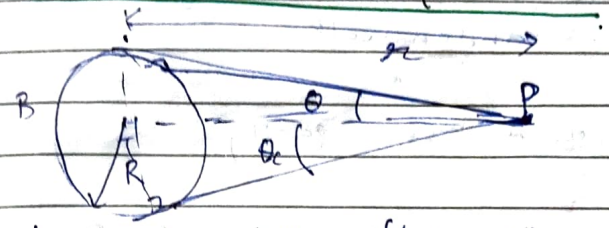
$$\int_{-1}^{1} u^2\,du = 2\int_{0}^{1} u^2\,du = 2\times\left(\frac{u^3}{3}\right)_0^1 = \frac{2}{3}$$

$$\rightarrow P_\nu = \frac{4\pi}{3}\frac{I_\nu}{c}$$

Now, $U_\nu = \frac{4\pi}{c} I_\nu$

$$P_\nu = \frac{1}{3} U_\nu$$ for isotropic Radiation.

Proving Inverse square Law.



Let us calc. flux at an arbitrary distance from a sphere of uniform brightness ($I_\nu = B_\nu$).

Such a sphere is clearly isotropic source

At P, the specific Intensity is B if the ray intersects the sphere & 0 o/w.

$$F_\nu = \int I_\nu \cos\theta\, d\Omega = B_\nu \int_0^{2\pi} d\phi \int_0^{\theta_c} \cos\theta \sin\theta\, d\theta.$$

$$\theta_c = \sin^{-1}\left(\frac{R}{r}\right)$$

So, we have, $F_\nu = \pi B_\nu (1 - \cos^2\theta_c)$.

$$= \pi B_\nu \sin^2\theta_c$$

$$F_\nu = \frac{\pi B_\nu R^2}{r^2}$$

So, $F_\nu \sim \frac{1}{r^2}$ D.

10) We used the following formula during the derivation of $\frac{dT}{dr}$ for radiation dominated part of stellar structure.

$$\frac{dP_{rad}}{dr} = -\frac{\bar{\kappa}\rho}{c} F_{rad}$$

Radiation pressure decreases as we move radially outward from star's center.

This pressure gradient in radiation pressure produces a slight net movement of photons toward surface in random walk.

$\bar{\kappa}$ = Rosseland mean opacity.

Formulation of Radiative Transfer Eqn

The monochromatic spontaneous emission coefficient is defined as the energy emitted per unit time, per unit solid angle, per unit volume in frequency range $[\nu, \nu+d\nu]$

$$dE = (j_\nu d\nu)\, dV\, d\Omega\, dt$$

In general, it depends in the direction into which emission is taking place.

Sometimes we also define a quantity called emissivity (ϵ_ν) it is defined as the energy emitted spontaneously per unit frequency per unit time per unit mass.

$$dE = \left(\frac{\epsilon_\nu \rho}{}\right) d\nu\, dV \left(\frac{d\Omega}{4\pi}\right) dt$$

$\hookrightarrow$ fraction of energy in that solid angle.

$$\therefore \boxed{j_\nu = \frac{\epsilon_\nu \rho}{4\pi}}$$

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In going a distance ds, a beam of cross section dA travels through a volume

$$dV = dA\,ds$$

In that case we can write the total intensity added to beam by this emission as,

$$dI_\nu \, d\nu \, dt \, dA \, d\Omega \, dA\,ds = j_\nu \, d\nu \, dV \, d\Omega \, dt$$

$$\therefore \boxed{dI_\nu = j_\nu \, ds}$$

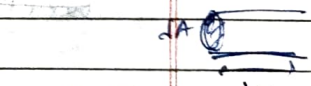
Now we define the Absorption coefficient (α_ν - cm^{-1})

which represents the loss of Intensity in a beam as it travels a distance ds we can write,

$$-dI_\nu \, d\nu \, dt \, dA \, d\Omega = (\alpha_\nu I_\nu) \, d\nu \, dA \, ds \, d\Omega \, dt$$

$$\boxed{dI_\nu = -\alpha_\nu I_\nu \, ds}$$

In general, if we think of a microscopic model in which particles with a number density(n) each present an effective absorbing area or cross section (σ_ν).



Then Total absorbing area = $N_{particle} \times \sigma_\nu$

$$= (n\,dV) \times \sigma_\nu$$

$$= n\,\sigma_\nu \, dA \cdot ds.$$

∴ The energy absorbed out of beam is.

$$-dI_\nu \, dA \, d\Omega \, dt \, d\nu = I_\nu (n\sigma_\nu \, dA\, ds) \, d\Omega \, dt \, d\nu$$

$$\boxed{dI_\nu = -\, n\,\sigma_\nu I_\nu \, ds}$$

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often

$$\alpha_\nu = \rho \kappa_\nu$$

ρ = Mass density

κ_ν $(cm^2 g^{-1})$ is called the Mass absorption coefficient, also called opacity coefficient.

$$\alpha_\nu = n \sigma_\nu = \rho \kappa_\nu.$$

Assumption of this model:

i. $\sqrt{\sigma_\nu} \ll d \sim n^{-1/3} \Rightarrow \alpha_\nu d \ll 1$

d → Inter particle distance.

ii. Absorbers are independent and randomly distributed.

Combining

$$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu + j_\nu.$$

This eqn takes a simple form, if we define,

$$d\tau_\nu := \alpha_\nu \, ds.$$

τ_ν = optical depth.

$$\tau_\nu(s) = \int_{s_0}^{s} \alpha_\nu(s') \, ds'$$

Along Ray path.

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A medium is

→ Optically Thick / opaque if $\tau_\nu > 1$

→ Optically Thin / Transparent if $\tau_\nu < 1$

The mean physical distance travelled in a homogenous medium is called mean free path.

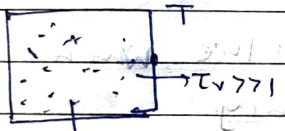
$$\left[l_\nu = \frac{1}{\alpha_\nu} = \frac{1}{n\sigma_\nu} \right]$$

$$\therefore \frac{dI_\nu}{\alpha_\nu\, ds} = -I_\nu + \underbrace{\frac{j_\nu}{\alpha_\nu}}$$

$$= S_\nu \quad \text{Source Function.}$$

We have, subst $\alpha_\nu = K_\nu \rho$

$$\boxed{\frac{1}{K_\nu \rho}\frac{dI_\nu}{ds} = -I_\nu + S_\nu} \quad \cdots (*_1)$$



optically thick gas in equilibrium → (LTE)

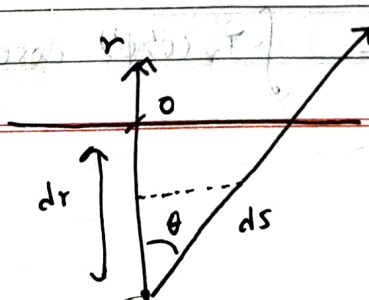
$\boxed{S_\nu = B_\nu(T)}$ Planck Black body Function.

$\Rightarrow dI_\nu = S_\nu \quad \forall \nu$

$\Rightarrow -1 \cdot \frac{dI_\nu}{ds} = 0$

(particle - photon equilibrium)

In eq $(*_1)$, 's' is the distance measured along the light ray's direction of travel.



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Write in radial coordinate.

$$ds \cos\theta' = dr$$

$$\therefore \frac{1}{K_\nu \rho} \frac{dI_\nu}{dr} \cos\theta' = -I_\nu + S_\nu$$

From the definition of Radiation pressure

Recall that,

$$P_{rad,\nu} = \frac{1}{c} \int I_\nu \cos^2\theta' \, d\Omega'$$

$$\therefore \frac{1}{K_\nu \rho} \frac{d}{dr} \int I_\nu \cos^2\theta' d\Omega' = -\int I_\nu \cos\theta' d\Omega' + \int S_\nu \cos\theta' d\Omega'$$

In order to ~~simplify~~ simplify analysis, we often replace K_ν with a constant value which is independent of frequency.

$K_\nu \rightarrow \overline{K}$ Rossland mean opacity

$$\overline{K_\nu} = \frac{\int_0^\infty \frac{1}{K_\nu} \frac{\partial B_\nu(T)}{\partial T} d\nu}{\int_0^\infty \frac{\partial B_\nu(T)}{\partial T} d\nu}$$

This model is called Gray Atmosphere.

Recall, definition of Radiative Flux

$$F_{\nu,rad} = \int I_\nu(\theta', \phi') \cos\theta' d\Omega'$$

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If we assume source function is isotropic then S_ν is a constant.

$$\therefore \int S_\nu(\theta', \phi') \cos\theta' \, d\Omega' \equiv S_{\nu,iso} \int \cos\theta' \, d\Omega'$$

$$= 0$$

$$\therefore -\frac{1}{k_\nu \rho} \frac{d P_{rad,\nu}}{dr} = F_{\nu,rad} + 0$$

(5) Integrating over all frequencies.

$$\Rightarrow \frac{c}{\rho} \frac{d}{dr} \int \frac{P_{rad,(\nu)}}{k_\nu} d\nu = -\int F_{rad}(\nu) \, d\nu$$

$$\Rightarrow \frac{c}{\rho \bar{k}} \frac{d}{dr} \int P_{rad,\nu} \, d\nu = -\int F_{rad,\nu} \, d\nu$$

$$\Rightarrow \boxed{\frac{d P_{rad}}{dr} = -\frac{\rho \bar{k}}{c} \vec{F}_{rad}}$$

(11) Assuming the opacity in very hot stars is provided by Thomson scattering, show that L/M has to be very less than a critical value and find its numerical value. How does this compare for sun?

Ans We know, $F_{rad} = \frac{L(r)}{4\pi r^2}$.

Now Pressure gradient due to radiation,

$$\frac{dP_{rad}}{dr} = -\frac{\rho \bar{k}}{c} \cdot \frac{L(r)}{4\pi r^2}$$

Also Pressure gradient due to gravity.

$$\frac{dP_g}{dr} = -\rho \frac{GM(r)}{r^2}$$

Now at the surface, $r = R_*$

$M(r = R_*) = M_*$

~~$L(r = R_*) = L_*$~~

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Hydrostatic equilibrium at the surface demands the Pressure gradients to be equal.

i.e $\frac{\bar{\kappa} \rho}{c} \frac{L}{4\pi r^2} \leq \rho \frac{G M_* \rho}{r^2}$

$$L_{Edd} = \frac{4\pi G c M_*}{\bar{\kappa}}$$

This is maximum luminosity that a star can have and still be in hydrostatic equilibrium, is called Eddington Luminosity Limit

if.

$L_* > L_{Edd} \Rightarrow$ Mass loss

In outer regions of massive stars, it is generally ionized hydrogens. scattering is due to the electrons with Thomson cross-section σ_T, while mass is set by protons.

$$\therefore \quad \bar{\kappa} = \sigma_T / m_p$$

General sources of Opacity

1. Bound-Bound Transition
2. Bound-free Absorption (Photoionization)
3. Free-free absorptions (Bremsstrahlung)
4. Electron scattering (Thomson scattering of photons)

$$\sigma_T = \frac{1}{6\pi\epsilon_0^2}\left(\frac{e^2}{m_e c^2}\right)^2 = 6.65 \times 10^{-29} \ m^2$$

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So,

$$\left(\frac{L_{Edd}^{*}}{M^{*}}\right) \leq \left(\frac{L_{Edd}}{M^{*}}\right) = \frac{4\pi G c m_p}{\sigma_T}$$

$$= \frac{4\pi \times 6.67\times10^{-11} \times 3\times10^{8} \times 1.67\times10^{-27}\ kg}{6.65\times10^{-29}\times 10}$$

$$= \frac{4\times\pi\times 3\times 1.67}{10}$$

$$= 6.292$$

$$\therefore \left(\frac{L^{*}}{M^{*}}\right) \leq 6.292 = \frac{L_{EDD}}{M} \quad W/kg$$

for Hydrostatic Equilibrium.

For Sun, $\frac{L_\odot}{M_\odot} = \frac{3.86\times10^{26}\ W}{2\times10^{30}}$

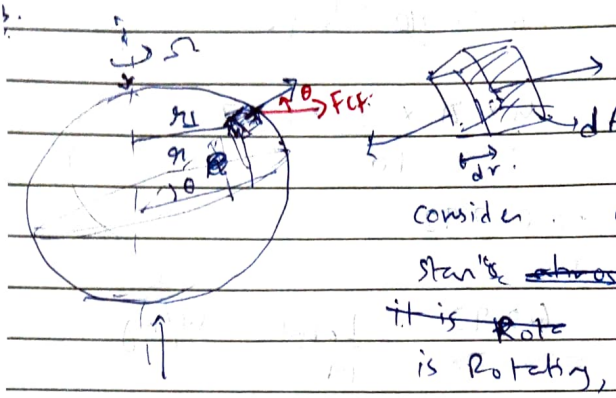
$= 1.93\times10^{-4} \ll 6.292$

Sun is safe!

$$\boxed{L_{EDD} \approx 3.26\times10^{4}\left(\frac{M^{*}}{M_\odot}\right)L_\odot}$$

However, quite different values of opacity, and hence Eddington Luminosity, are possible. Eg:- in low-mass cool stars opacity is given by Kramer's Law, $\kappa \propto \rho T^{-3.5}$, while in very hot environments, positron-electron plasma can be created with twice the scattering particles at $\sim 1/1000$ of the mass, ie L_{EDD} ~2000 times.

(8) In the class we derived the hydrostatic equilibrium by considering gravity and pressure gradient. Consider a star is also rotating with a constant angular velocity Ω about rotation axis. Is it possible to develop the equation of hydrostatic equilibrium for this configuration?

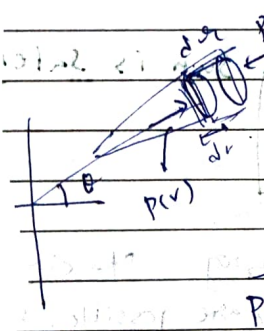


consider a block of star's ~~atmosphere~~, if star ~~it is Rote~~ is Rotating, the part will ~~also~~ also experience outward centrifugal force

$$dF_{cent} = dm \times \Omega^2 r_\perp$$

$$= \rho(\bar{r}) \, dr \, dA \, \Omega^2 r \cos\theta$$

$$= \rho(\bar{r}) \, \Omega^2 r \cos\theta \, dr \, dA$$



while there will be pressure gradient as usual due to gravity

The Net Acceleration of cylindrical gas element.

$$dm = (\text{Force pushing outward}) - (\text{Force pulling Inward})$$

$$\rho \, dr \, dA = [P(r) - P(r+dr)] \, dA + \rho \Omega^2 r \cos\theta \cos\theta \, dr \, dA - g(r) \rho \, dr \, dA$$

Radially outward component

$$= -\frac{\partial P}{\partial r} dr$$

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$$\ddot{r} = -\frac{1}{\rho}\frac{\partial P}{\partial r} + \Omega^2 r\cos^2\theta - \frac{GM(r)}{r^2}$$

For equilibrium $\ddot{r} = 0$

$\rho \equiv \rho(\vec{r}, t)$.

$$\therefore \quad \frac{\partial P(r,\theta)}{\partial r} = \rho\Omega^2 r\cos^2\theta - \rho\frac{GM(r)}{r^2}$$

(9) Assuming the Sun has radiated away half of its gravitational energy content, show that it's lifetime comes out to be around 10^7 years.

Kelvin Helmholtz Timescale.

The grav. potential energy of a test mass dm outside a M(r) shell is.

$$dU_g = -\frac{GM(r)}{r}\,dm$$

$$= -\frac{GM(r)}{r}\cdot 4\pi r^2 \rho(r)\,dr$$

Integrating over all shells from centre of star to surface, we find, it gravitational pot. energy,

$$U_G = -4\pi G \int_0^{R_*} M(r)\,\rho(r)\,r\,dr.$$

We can approximate this by.

$$\rho \sim \bar{\rho} = \frac{M_*}{\frac{4}{3}\pi R^3} \rightarrow M_* = \text{Total stellar Mass}$$

$$M(r) \sim \frac{4}{3}\pi r^3 \bar{\rho}$$

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$$U_G = -4\pi G \int_0^R \frac{4\pi \bar{\rho}^2}{3} r^4 \cdot dr$$

$$= -\frac{16\pi^2 \bar{\rho}^2 G R^5}{15}$$

Subst. $\bar{\rho} = \frac{M}{\frac{4}{3}\pi R^3}$.

$$\approx -\frac{16\pi^2}{15\pi} \cdot \frac{M^2}{\frac{16\pi^2}{9}} \frac{G}{R^3}$$

$$\therefore U_G \sim -\frac{16\pi^2 G \bar{\rho}^2 R^5}{15}$$

$$\sim -\frac{3}{5}\frac{GM^2}{R}.$$

Now, acc. to virial Theorem, a gravitational system moving from one equilibrium state of another will release ½ it's available energy as internal kinetic

So, Assuming sun were originally larger today, with $R_i \gg 1 R_\odot$

Then exactly ½ it's internal energy was radiated away during collapse.

$$\Delta E_g = -(E_f - E_i)$$

$$\simeq -E_f \simeq -\frac{U_G}{2} = \frac{3}{10}\frac{GM_\odot^2}{R_\odot}$$

$$\simeq 1.1 \times 10^{41} J$$

Assuming also that luminosity of sun has been roughly constant throughout its lifetime.

$$t_{KH} \simeq \frac{\Delta E_g}{L_\odot}$$

$$\simeq \frac{1.1 \times 10^{41}\ J/m^2\ s}{3.86 \times 10^{26}}$$

$$\sim 10^{14}\ \text{years}.$$

$$\sim \frac{1}{86400 \times 365} \times 10^{14}\ \text{yrs}.$$

$$\sim 10^7\ \text{years}.$$

□

Q.4) Sirius is a visual binary with a period of 49.94 yrs. Its measured trigonometric parallax is $(0.37921 \pm 0.00158)''$

~~Assume $i = 0$~~ (

Assume, angle of orbit inclination = 0,

The true angular extent of semi-major axis of reduced mass is 7.6".

The ratio of distances of sirius A and Sirius B from COM is $a_A/a_B = 0.466$.

(a) Find the mass of each member of the system

(b) $M_A = 1.36$, $M_B = 8.79$. (Bolometric) Determine their luminosities, in terms of $L_\odot$.

(c) For sirius B, $T_B = (24790 \pm 100)\ K$

Estimate it's radius and compare wrt. $\left(\frac{R_\odot}{R_\oplus}\right)$

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We know from Kepler Laws.

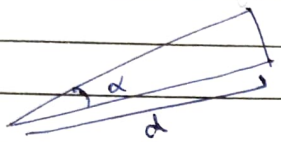
(a)

$$T_{orb}^2 = \frac{4\pi^2}{G(M_A+M_B)} a^3$$

$$a = a_A + a_B$$

And $M_A a_A = a_B M_B$

Put, $T = 49.97$ yrs.



Now

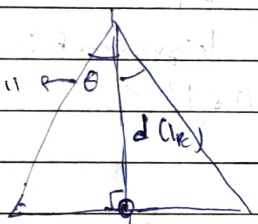
$$a = d\alpha$$

$d = ?$

Given

$\theta = 0.37921''$

We know



Recall,

$$1\,(pc) = \frac{1\,AU}{1''}$$

$$\tan\theta \simeq \frac{1}{d}$$

$$d \simeq \frac{1}{\theta}$$

Now,

$$\therefore d(pc) = \frac{1\,AU}{\theta''}$$

$$= \left(\frac{1}{0.37792}\right) \frac{1\,AU}{1''}$$

$$= \left(\frac{1}{0.37792}\right) \cdot 1\,pc$$

$$= 2.637\ pc$$

True.

∴ length of semi major axis

$$a = 7.61 \times \frac{\pi}{180} \times \frac{1}{3600} \times 5.294$$

rad.

$$\triangleq \frac{1.946 \times 10^{-4}\ pc}{2}$$

Now, conversion

$$1AU \cong 1.5 \times 10^{11}\ m$$

And $$1\,(pc) = \left(\frac{1}{\frac{1}{3600} \times \frac{\pi}{180}}\right) \frac{AU}{rad}$$

$$\triangleq 206265\ AU$$

$$\therefore a = 1.946 \times 10^{-4} \times 206265 \times 1.5 \times 10^{11}\ m$$

$$= \frac{6.02 \times 10^{12}\ m}{2} = 3.01 \times 10^{12}\ m$$

Now,

Time period

$$T_{orb} \triangleq 49.94 \times 365.25 \times 86400$$

$$\simeq 1.5759.865.44\ s$$

$$= 1.576 \times 10^{9}\ s$$

$$\therefore M_A + M_B = \frac{4\pi^2}{G} \frac{a^3}{T_{orb}^2}$$

$$= \frac{4\pi^2}{6.67 \times 10^{-11}} \frac{(3.01 \times 10^{11})^3}{(1.576)^2 \times 10^{18}}\ 10^{33}\ kg$$

$$= 5.198 \times 10^{31}\ kg$$

$$6.4986 \times 10^{30}\ kg$$

$$1\ M_\odot = 2 \times 10^{30}\ kg$$

$$\therefore M_A + M_B \triangleq 3.25\ M_\odot$$

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$$\frac{M_A}{M_B} = 0.466 = \frac{a_B}{a_A}$$

$$M_A + M_B = 3.25\ M_\odot$$

$$M_B\left(1 + \frac{a_B}{a_A}\right) = 3.25$$

$$\therefore M_B = \frac{3.25}{1 + 0.466^{-1}} \simeq 1.033\ M_\odot$$

$$\therefore M_A = \frac{}{} = 2.216\ M_\odot$$

(b)

$$m = M + 5\log_{10}\left(\frac{d}{10\,pc}\right)$$

Now, Finding Apparent magnitude.

$$m_A = M_A + 5\log_{10}\left(\frac{d_A}{10}\right)$$

$$= 1.36 + 5\log_{10}\left(\frac{2.637}{10}\right)$$

$$= 1.36 - 2.894 = -1.534$$

$$m_B = 8.79 + 5\log_{10}\left(\frac{2.637}{10}\right)$$

$$= 5.8955$$

Now, apparent magnitude is defined as

$$m = -2.5 \log_{10}(f) + C$$

(f → flux; C → constant)

$$m_\odot = -2.5 \log_{10}\left(\frac{f_\odot}{4\pi}\right) + C$$

Now,

$$f_\star = \frac{L}{4\pi d^2}$$

$$\therefore m - m_\odot = -2.5 \log_{10}\left(\frac{f_\star}{f_\odot}\right)$$

$$= -2.5 \log_{10}\left(\frac{L}{L_\odot}\right) + 5 \log_{10}\left(\frac{d}{d_\odot}\right)$$

For sun

$m_\odot = -26.83$

$d_\odot = 1/206265$ pc

$$\therefore 2.5 \log_{10}\left(\frac{L}{L_\odot}\right) = 5 \log_{10}\left(\frac{d}{d_\odot}\right) + (m_\odot - m)$$

For Sirius A

$$= 5 \log_{10}\left(2.637 \times 206265\right) + (m_\odot - m_A)$$

$$= 28.677 + (-26.893 + 1.534)$$

$$= 30381$$

$$\therefore L_A = 22.52\, L_\odot$$

$$2.5 \log_{10}\left(\frac{L_B}{L_\odot}\right) = 28.677 + (-26.83 - 5.895)$$

$$\Rightarrow$$

$$\boxed{L_B = 0.024\, L_\odot}$$

(c) From Stefan Boltz law

$$L_B = 4\pi R_B^2 \sigma T_B^4$$

$$R_B^2 = \frac{L_B}{4\pi\sigma T_B^4}$$

$$R_B = \left(\frac{0.024 \times 3.86 \times 10^{26}}{4 \times \pi \times 5.67 \times 10^{-8} \times 24790^4}\right)^{1/2}$$

$$= 5867455 \text{ m}$$

$$R_B = 5867.45 \text{ km}.$$

$$\therefore R_B = \left(\frac{5867.45}{6400}\right) R_\oplus$$

$$\boxed{R_B = 0.9168\, R_\oplus}$$

$$= 0.0084\, R_\odot$$

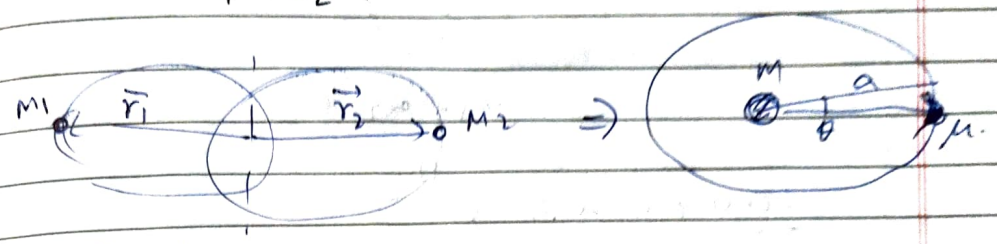
Q2) Consider two stars in orbit about a mutual center of mass.

If a_1 = semi major axis of m_1 orbit
a_2 = " " " m_2.
a = " " " reduced mass

P.T

$\mu = \frac{m_1 m_2}{m_1 + m_2}$

$a = a_1 + a_2$.



Recall

$\vec{r} = \vec{r}_1 - \vec{r}_2$

And S.t $\vec{R}_{CM}$ in COM frame.

$\vec{R}_{CM} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2} = 0$

$\rightarrow m_1 \vec{r}_1 = -m_2 \vec{r}_2$.

$\rightarrow \frac{m_1}{m_2} = \frac{r_2}{r_1} = \frac{a_2}{a_1}$

Since, $r_1 = \frac{a_1(1-e^2)}{1 + e\cos\theta}$ (magnitude wise) $\therefore$

Since, $\vec{r}_1 \parallel \vec{r}_2$ are oppositely directed relative to COM,

~~$r = r_1 + r_2$~~

Separation b/w m_1 & m_2 is r

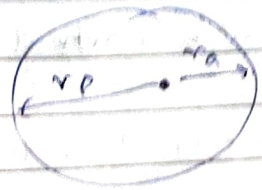
actually, $r = r_1 + r_2$

$\therefore r = \frac{(a_1 + a_2)(1-e^2)}{1 + e\cos\theta}$.

From this result, $r_p = r(\theta = 0) = (a_1 + a_2)(1-e)$

$r_a = r(\theta = \pi) = (a_1 + a_2)(1+e)$

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$\therefore r_a + r_p = 2a$

a: $\frac{r_a + r_p}{2} = \frac{(a_1 + a_2)(1-e) + (a_1 + a_2)(1+e)}{2}$

$= a_1 + a_2$

Hence proved. ∎

M-2

Recall in COM frame

$m_1 \vec{r}_1 = -m_2 \vec{r}_2$

$\therefore \frac{m_1}{m_2} = \left| \frac{-\vec{r}_2}{\vec{r}_1} \right| = \frac{r_2}{r_1}$

When r_2 greatest, r_1 also greatest

for $\frac{m_1}{m_2}$ to be constant.

∴ When star 1 is at aphelion, star 2 will also be at "

$\vec{L} = \mu \vec{r} \times \vec{v}$

$= \mu |r| \hat{r} \times \frac{d}{dt}(r\hat{r})$

$= \mu r \hat{r} \times \left[\frac{dr}{dt} \hat{r} + r \frac{d\hat{r}}{dt} \right]$

$= 0$

$\vec{L} = \mu r^2 \hat{r} \times \frac{d(\hat{r})}{dt}$

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Now, $M = m_1 + m_2$

$$\vec{a} = -\frac{GM}{r^2}\hat{r}$$

$$\vec{a} \times \vec{L} = -GM \hat{r} \times \left(\mu r^2 \hat{r} \times \frac{d}{dt}\hat{r}\right)$$

$$= -GM\mu \hat{r} \times \left(\hat{r} \times \frac{d\hat{r}}{dt}\right)$$

$$\vec{a} \times \vec{L} = -GM\mu \left[\left(\hat{r} \cdot \frac{d\hat{r}}{dt}\right)\hat{r} - \frac{d\hat{r}}{dt}(\hat{r}\cdot\hat{r})\right]$$

$\left(\hat{r} \cdot \frac{d\hat{r}}{dt}\right) = 0$ $\because \frac{d\hat{r}}{dt} = \dot{\theta}\hat{\theta}$, $\hat{r}\cdot\hat{\theta} = 0$

$(\hat{r}\cdot\hat{r}) = 1$

$$\Rightarrow \vec{a} \times \vec{L} = +GM\mu \frac{d\hat{r}}{dt}$$

$$\Rightarrow \frac{d\vec{v}}{dt} \times \vec{L} + \vec{v} \times \frac{d\vec{L}}{dt} = \frac{d}{dt}(GM\mu\hat{r})$$

$\frac{d\vec{L}}{dt} = 0$ $\because$ in central Force

$$\Rightarrow \frac{d}{dt}(\vec{v} \times \vec{L}) = \frac{d}{dt}(GM\mu\hat{r})$$

$$\Rightarrow \vec{v} \times \vec{L} = GM\mu\hat{r} + \vec{A}$$ ($\vec{A}$: constant)

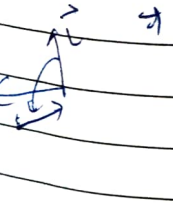
$\vec{A}$ is a vector in orbital plane.

$$\boxed{\vec{A} = \vec{v} \times \vec{L} - GM\mu\hat{r}}$$

$= \text{constant}$

Now $\vec{r} \cdot (\vec{v} \times \vec{L}) = \vec{r} \cdot (GM\mu\hat{r}) + \vec{A} \cdot \vec{r}$ (A · B × C = C · (A × B) = B · (C × A))

$\vec{L} \cdot (\vec{r} \times \vec{v}) = GM\mu r + Ar\cos\theta$



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subst.

$$\vec{L} = \mu \vec{r} \times \vec{v}$$

$$\therefore \frac{\vec{L}^2}{\mu} = GM\mu r + Ar\cos\theta$$

$$= GM\mu r\left(1 + \frac{A\cos\theta}{GM\mu}\right)$$

$$\Rightarrow r = \frac{\vec{L}^2}{GM\mu^2} \frac{1}{(1 + e\cos\theta)}$$

$$\boxed{e = \frac{A}{GM\mu}}$$

$$r = r_1 + r_2. \qquad \frac{r_1}{r_2} = \frac{m_2}{m_1}$$

$$= r_2\left(1 + \frac{m_2}{m_1}\right) \qquad r_1 = \frac{m_2}{m_1} r_2$$

$$= r_2\left(\frac{m_1 + m_2}{m_1 m_2}\right) m_2$$

$$= \frac{r_2 m_2}{\mu}$$

$$\rightarrow r_2 = \frac{\mu r}{m_2}$$

$$r_1 = \frac{\mu r}{m_1}$$

$$\therefore r_1 = \frac{\vec{L}^2}{GMm_1\mu} \frac{1}{(1 + e\cos\theta)}$$

$$r_2 = \frac{\vec{L}^2}{GMm_2\mu} \frac{1}{1 + e\cos\theta}$$

Both stars have same eccentricity of orbit, & reduced mass.

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∴ At Aphelion,

$$\vec{r_1} = -a_1(1+e)\hat{n}$$

$$\vec{r_2} = a_2(1+e)\hat{n}$$

$$\vec{r} = \vec{r_2} - \vec{r_1}$$

$$= (a_1 + a_2)(1+e)$$

$$= a(1+e)$$

$$\frac{dP}{dr} = -\rho \frac{Gm}{r^2}$$

$$\frac{dP}{dm} = -\frac{Gm}{4\pi r^4}$$

$$\frac{m}{4\pi r^2} = \rho$$

Newton's Law of gravity, tells that.

Force on outer shell is.

$$F = G M_1 M_2 / r^2 \sim G M^2 / R^2$$

$$\therefore \text{Pressure} \sim \frac{F}{4\pi R^2} \sim \frac{F}{R^2}$$

$$\therefore P_c R^2 \sim G M^2 / R^2$$

Or $$\boxed{P_c \sim \frac{G M^2}{R^4}}$$

Core Temperature : $P = k_B n T$ Equation of state.

$$P_c \cong k_B n_c T_c$$

The no. of atoms in the sun $= \frac{M_\odot}{m_p} = N$ m_p = mass of proton

Avg. density of atoms, $n = \frac{N}{\frac{4}{3}\pi R_\odot^3} \sim \frac{M_\odot}{R_\odot^3 m_p}$

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$$K_B n_c T_c \sim \frac{GM^2}{R^4}$$

$$K_B \frac{M}{R^3 m_p} T_c \sim \frac{GM^2}{R^4}$$

$$T_c \sim \frac{GMm_p}{K_B R}$$

Q) Derive the formal solution of Radiative Transfer equation and discuss the physical significance of each term of the solution.

Recall

$$\frac{dI_\nu}{d\nu}\frac{d\nu}{ds} = -I_\nu + \frac{j_\nu}{\alpha_\nu}$$

$$\frac{dI_\nu}{d\tau_\nu} = -I_\nu + S_\nu$$

I_ν = incident brightness

S_ν = Source function → result of stimulated emission

$\tau_\nu = \int \alpha_\nu(z)\, ds'$ is optical depth.

$\frac{dI_\nu}{d\tau_\nu}$ is rate of change of Intensity with optical depth.

$$\frac{dI_\nu}{d\tau_\nu} + I_\nu = S_\nu$$

I.F $= e^{\int d\tau_\nu} = e^{\tau_\nu}$

$$e^{\tau_\nu}\frac{dI_\nu}{d\tau_\nu} + e^{\tau_\nu} I_\nu = S_\nu \cdot e^{\tau_\nu}$$

$$\frac{d}{d\tau_\nu}\left(e^{\tau_\nu} I_\nu\right) = S_\nu e^{\tau_\nu}$$

$$\int_{\tau_\nu=0}^{\tau_\nu} d\left(e^{\tau'_\nu} I_\nu\right) = \int_0^{\tau_\nu} S_\nu e^{\tau'_\nu} d\tau'_\nu$$

$$I_\nu e^{\tau_\nu} = I_{\nu_0} + \int_0^{\tau_\nu} S_\nu e^{\tau'_\nu} d\tau'_\nu$$

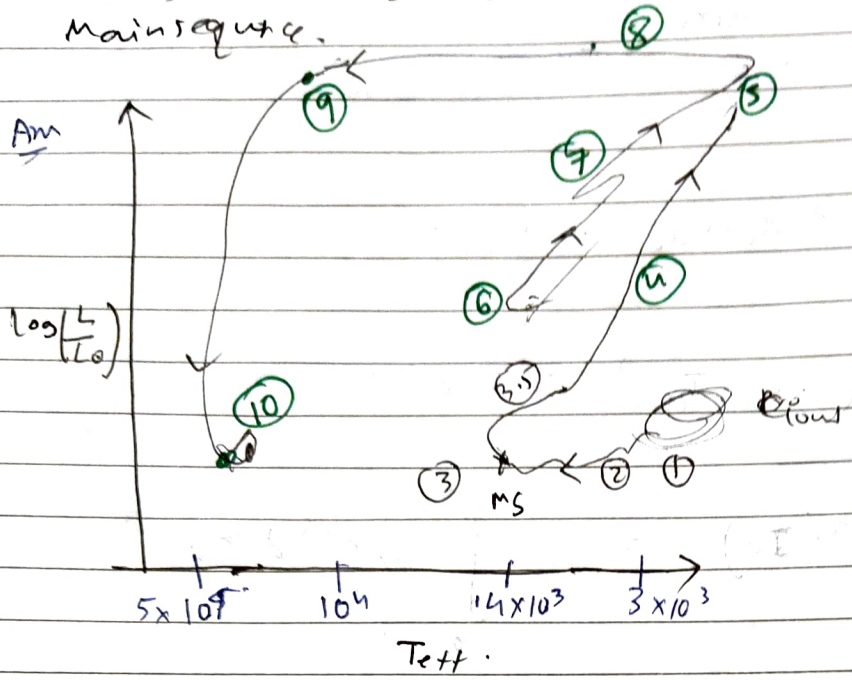
I_{ν_0} : $I_\nu(\tau_\nu = 0)$

$$I_\nu(\tau_\nu) = I_{\nu_0} e^{-\tau_\nu} + \int_0^{\tau_\nu} S_\nu e^{-(\tau_\nu - \tau'_\nu)} d\tau'_\nu$$

This is the original diminished intensity by $e^{-\tau_\nu}$

Integrated source diminished by exponential factor.

(5) Briefly describe the evolution of a 1 M⊙ star since its beginning of lifetime on mainsequence.



For a typical ~~sun~~ yellow like dwarf star,

The typical phase is like.

(1) Dense Giant Molecular Cloud.

↓ Gravitational collapse

(2) Protostar.

↓

(3) Main Sequence — H-burning (Most of its lifetime)

3,5) Sub Giant branch.

(4) Red giant Branch. → shell burning of H

(5) Helium Flash

(6) Red Clump ---→ He Burning in core

(7) Asymptotic Giant Branch (AGB) → outer shell → O, C burning; shell burning of He.

(8) Post AGB → Nebula → Bare remain, Oxygen, Carbon

(9) Planetary

(10) White Dwarf.

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Q.6 Virial Theorem for stars states the that total Energy ~~$E_{int} = \Omega$~~.

$$E_{int} = -\frac{1}{2}\Omega_*$$

Ans Let's start with HE

Assumption

- Stars are isolated in space, No binary systems
- Spherically symmetric stars.
- LTE

Now $dm = 4\pi r^2 \rho(r)\, dr$

$$\frac{dP}{dr} = 0 - \frac{\rho_{in} G M}{r^2}$$

$$4\pi r^2 \rho(r) \cdot \frac{dP}{dm} = -\rho(r) \frac{GM}{r^4}$$

$$\boxed{\frac{\partial P}{\partial m} = -\frac{Gm}{4\pi r^4}}$$

$$\int_0^{M_*} dm \cdot \frac{4\pi}{3} r^3 \frac{dP}{dm} = \int_0^{M_*} -\frac{Gm}{4\pi r^4} \cdot \frac{4}{3}\pi r^3\, dm$$

$$\int_{P_c}^{P(R_*)} V\,dP = \left(VP\right)_{P_c}^{P(R_*)} = -\int_0^{V_*} P\,dV$$

$$P(R_*) = 0 \text{ (hydeqm)}$$

And At center, $V = 0$.

$$\therefore \text{Ef}$$

$$= -\frac{1}{3}\int_0^{M_*} \frac{Gm}{r}\, dm \equiv \frac{1}{3}\Omega_*$$

Ω_* ← grav. potential energy

$$\Omega_* = -\int_0^{M_*} \frac{GM}{r}\, dm$$

Thm:

$$-\int_0^{V_*} P\,dV = \Omega_*/3$$

$$dV = \frac{dm}{\rho}$$

$$\therefore \Omega_* = -3\int_0^{M_*} \frac{P}{\rho}\,dm$$

For Monoatomic Ideal gas. (Assume)

Eqn of state | $P = nK_BT$

$$P = \frac{N}{V}K_BT$$

$$P = \frac{\rho}{(M/N)}K_BT$$

$$\rho = \frac{M}{V}$$

$$\frac{1}{V} = \frac{\rho}{M}$$

$$P = \frac{\rho}{\mu m_p}K_BT$$

$M = N \times \mu \times m_p$

- N: Total No. of molecules
- μ: Mean mol. weight
- m_p: mass of proton

Now, Total Internal Energy per molecule,

$$u \times \mu m_p = \frac{3}{2}K_BT$$

Energy density per unit mass of star

$$u = \frac{3}{2}\frac{K_BT}{\mu m_p}$$

$$u = \frac{3}{2}\frac{P}{\rho}$$

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$$\therefore \Omega_* = -\frac{3}{5}\frac{2}{3}\int_0^{M_*} u\, dm$$

$$= -2 E_{int}$$

E_{int} = Total internal energy of star

(b) In general, a star is made up of hydrogen, helium and metals having mass fraction. ~~X and Y and~~ X, Y and Z:
Derive an expression for mean molecular weight μ that can be used to calculate the pressure for such a star.

$P(\rho, T, X_i) \rightarrow$ Equation of state.

$$X \equiv \frac{\rho_H}{\rho}, \quad Y = \frac{\rho_{He}}{\rho}, \quad Z = \frac{\rho_{metals}}{\rho}$$

7 coupled eqn → 7 unknowns

$P, M, \rho, T, R, L, \epsilon$

4 boundary condition → sol^n is unique
for 4 ODE ↓
Vogt-Russel conjecture

In most normal stars, ~~the~~ under conditions in most normal stars, the equation of state of a classical, non-relativistic, ideal gas provides a good description.

The mean particle mass will be,

$$\bar{m} = \frac{n_1 m_1 + n_2 m_2 + n_3 m_3}{n_1 + n_2 + n_3} = \frac{\rho}{n}.$$

n_i = density of i^{th} type particle
m_i = mass " " " " .
ρ = mass density.

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∴ Gas pressure,

$$P_g = n K_B T$$

$$\boxed{P_g = \frac{\rho}{\bar{m}} K_B T}$$

Now, Mean mass will depend on chemical abundance and ionization state of gas.

For completely ionized hydrogen $(e^- + p)$

$$\bar{m} = \frac{m_H}{2} \quad e^- + p^+ \rightarrow$$

More generally, the no. densities of hydrogen, helium or an element of atomic mass Number A.

i.e., A = Total No. of Nucleon = p + n.

Z = No. of p. (e^-).

Z_A = Mass Abundance of element

$$n_A = Z_A \cdot \frac{\rho}{A m_H} \cdot Z$$

$2e^- + 1$ Nucleon

$(1e^- + 1p)$

$$n_H = \frac{2\rho}{m_H}, \quad n_{He} = \frac{3\rho}{4 m_H}$$

$(2n + 2p)$

Basically,

$$n_A = \frac{A}{2A} \frac{\rho}{m_H}$$

$$= \frac{1}{2} \frac{\rho}{m_H}$$

$Z_A \sim Z + 1$

$\simeq A/2$

For heavy atom.

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∴ For an ionized gas, ∴

$$n = X \cdot n_H + Y \cdot n_{He} + Z \cdot n_{metals}$$

∴

No. of Ionized species.

$$n_H = \frac{2 \times \rho}{1 \times m_H}$$

$$n_{He} = \frac{3 \times \rho}{4 \times m_H}$$

$$n_{metal} = \frac{b}{2A} \frac{\rho}{m_H}$$

No. of Nucleon

$$\simeq \frac{1}{2} \times \frac{\rho}{m_H}$$ Most Cases.

$$\therefore n = \frac{\rho}{m_H}\left(2X + \frac{3Y}{4} + \frac{Z}{2}\right).$$

$$n = \frac{\rho}{2m_H}\left(3X + \frac{Y}{2} + 1\right)$$

$Z = 1 - X - Y$

$$P = \frac{n K_B T}{\text{gas}} + \frac{1}{3} a T^4$$ (radiation)

Mean molecular weight $= \bar{m}$

$$\bar{m} = \frac{\rho}{n} = m_H \times \left(\frac{2}{3X + 0.5Y + 1}\right)$$

$X_\odot = 0.71$

$Y_\odot = 0.27$ $\therefore \bar{m}/m_H \to 0.61$

$P_g = \frac{K_B T}{\bar{m}}$ ∴ $M = \frac{\bar{m}}{m_H}$

#

Energy Transport Equation.

(a) Radiative Part

$$\frac{dP_{rad}}{dr} = -\frac{\bar{\kappa}\rho}{c} F_{rad}$$

Now $F_{rad} = \frac{L}{4\pi r^2}$

$$P_{rad} = \frac{1}{3} a T^4$$

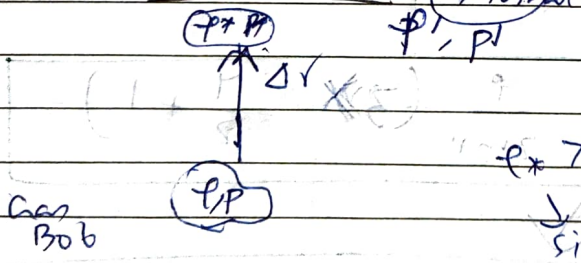
For radiation

$$\frac{\partial P_{rad}}{\partial r} = \frac{4}{3} a T^3 \frac{\partial T}{\partial r}$$

$$\frac{4}{3} a T^3 \frac{\partial T}{\partial r} = -\frac{\bar{\kappa}\rho}{c} \frac{L}{4\pi r^2}$$

$$\Rightarrow \frac{\partial T}{\partial r} = -\frac{3\bar{\kappa}\rho}{16\pi a c} \frac{L(r)}{r^2 T^3}$$

(b) Convection Part



$PV^\gamma = \text{const.}$

~~$V \propto \rho^\gamma$~~ $\rho = \frac{m}{V}$

$V \sim \frac{1}{\rho}$

$P\rho^{-\gamma}$ const.

$\therefore \rho \sim P^{1/\gamma}$

$$\left(\frac{\rho^*}{\rho}\right) = \left(\frac{P^*}{P}\right)^{1/\gamma}$$

As $P^* = P'$ (For equilibrium of pressure)

$$\rho^* = \rho\left(\frac{P'}{P}\right)^{1/\gamma}$$

$$P' = P + \frac{\partial P}{\partial r}\Delta r.$$

$$\rho^* = \rho\left(1 + \frac{1}{P}\frac{\partial P}{\partial r}\Delta r\right)^{1/\gamma}$$ ← density of blood.

-ve

Small.

$$\rho^* \simeq \rho + \frac{\rho}{\gamma P}\frac{\partial P}{\partial r}\Delta r$$

on other hand external density will increase? acc. to.

$$\rho' = \rho + \frac{\partial \rho}{\partial r}\Delta r.$$ Normal External density

Now, using $\rho = \frac{P}{RT}$

$$\frac{d\rho}{\rho} = \frac{dP}{P} - \frac{dT}{T} + 0$$

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$$\rho' = \rho + \frac{\rho}{P}\frac{\partial P}{\partial r}\Delta r - \frac{\rho}{T}\frac{\partial T}{\partial r}\Delta r$$

$$\therefore \rho^* - \rho' = \rho + \frac{\rho}{P}\frac{\partial P}{\partial r}\Delta r - \rho - \frac{\rho}{P}\frac{\partial P}{\partial r}\Delta r + \frac{\rho}{T}\frac{\partial T}{\partial r}\Delta r$$

$$\rho^* - \rho = \left(-\frac{\rho}{P}\frac{dP}{dr}\left(1-\frac{1}{\gamma}\right) + \frac{\rho}{T}\frac{dT}{dr}\right)\Delta r$$

Criteria

For stable Atmosphere. i.e.

is $\rho^* - \rho' > 0 \Rightarrow$ No convection

$$-\frac{\rho}{P}\frac{dP}{dr}\left(1-\frac{1}{\gamma}\right) + \frac{\rho}{T}\frac{dT}{dr} > 0$$

Now $\frac{dP}{dr} = -\left|\frac{dP}{dr}\right|$

$\frac{dT}{dr} = -\left|\frac{dT}{dr}\right|$

$$\therefore \boxed{\left|\frac{dT}{dr}\right| < \left(1-\frac{1}{\gamma}\right)\frac{T}{P}\left|\frac{dP}{dr}\right|}$$

Schwarzschild stability criterion.

Q 20)

We derived Euler's Equation of fluid dynamics in class.

$$\frac{d\vec{v}}{dt} = -\frac{1}{\rho}\vec{\nabla}P + \frac{\vec{f}}{\rho}$$

$\vec{f} = \rho\vec{g}$

$= \vec{g}$ $\qquad$ $= -\rho\nabla\Phi$

From here derive Bernoulli's Principle for steady flow.

$$\frac{1}{2}v^2 + \int\frac{dp}{\rho} + \Phi = \text{constant}.$$

Φ = gravitational potential.

$\vec{g} = -\nabla\Phi$

Φ = energy/mass.

Now,

$$\frac{d\vec{v}}{dt} = \frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v}$$

Now

steady state $\Rightarrow$ quantities dont change with time. A-1

i.e $\frac{\partial\vec{v}(\vec{x})}{\partial t} = 0$.

$$\therefore (\vec{v}\cdot\nabla)\vec{v} = -\frac{1}{\rho}\nabla P + -\nabla\Phi$$

for irrotational flow A-2 $\vec{\nabla}\times\vec{v} = 0$.

Now $\nabla\times(\nabla\times v) =$

$\downarrow$ $\qquad$ $\frac{1}{2}\vec{\nabla}v^2 - (\vec{v}\cdot\nabla)\cdot\vec{v}$

0

$$\therefore (\vec{v}\cdot\nabla)\vec{v} = \frac{1}{2}\vec{\nabla}v^2$$

Replace.

$$\vec{\nabla}\left(\frac{v^2}{2}\right) + \frac{1}{\rho}\nabla P + \vec{\nabla}\Phi = 0.$$

$$\int\vec{\nabla}\left(\frac{v^2}{2}\right)\cdot d\vec{l} + \int\frac{\nabla P\cdot d\vec{l}}{\rho} + \int\nabla\Phi\cdot d\vec{l}$$

Integrate along the particle fluid trajectory.

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$$\left[\frac{v^2}{2} + \int\frac{dP}{\rho} + \Phi = \text{constant}\right]$$

Q.

For incompressible fluid, $\nabla \rho = 0$. $\frac{1}{\rho}\nabla P = \nabla\left(\frac{P}{\rho}\right)$

$$\nabla\left(\frac{v^2}{2} + \frac{P}{\rho} + \Phi\right) = 0$$

$$\int \nabla(\,) \cdot dl = 0$$

$$\Rightarrow \boxed{\frac{v^2}{2} + \frac{P}{\rho} + \Phi = \text{constant}}$$

Q) Show Mass continuity & Euler's Equation can be combined to give,

$$\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j}(\rho v_i v_j + P\delta_{ij}) = \rho F_i$$

Ans We have

(i) Eq. of continuity

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \bar{v}) = 0$$

$$\rightarrow \frac{\partial \rho}{\partial t} + \frac{\partial(\rho v_k)}{\partial x_k} = 0$$

(ii) Euler's Equation with external forces become —

$$\frac{\partial \bar{v}}{\partial t} + (\bar{v}\cdot\nabla)\bar{v} = \frac{-1}{\rho}\nabla P + \bar{f}$$

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$$\frac{\partial v_i}{\partial t} + (\bar{v}\cdot\nabla) v_i = \frac{-1}{\rho}\frac{\partial p}{\partial x_i} + f_i$$

$$\boxed{\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} = \frac{-1}{\rho}\frac{\partial p}{\partial x_i} + f_i} \quad \ldots (ii)$$

Now,

$$\frac{\partial}{\partial x_k}(\rho v_i v_k) = v_i \frac{\partial(\rho v_k)}{\partial x_k} + \rho v_k \frac{\partial v_i}{\partial x_k}$$

Subst this.

$$\boxed{\frac{\partial}{\partial x_k}(\rho v_i v_k) = \rho v_k \frac{\partial v_i}{\partial x_k} - v_j \frac{\partial \rho}{\partial t}}$$

from eq (i) Now, to replace $\frac{\partial \rho}{\partial t}$, $\frac{\partial v_i}{\partial t}$

We have.

$$\frac{\partial(\rho v_i)}{\partial t} = v_i \frac{\partial \rho}{\partial t} + \rho \frac{\partial v_i}{\partial t}$$

Subst. in eq (ii)

Substitute

$$\frac{\partial}{\partial t}(\rho v_i) = v_i \frac{\partial \rho}{\partial t} + \rho\left[\frac{-1}{\rho}\frac{\partial p}{\partial x_i} + f_i - v_j \frac{\partial v_i}{\partial x_j}\right]$$

$$= v_i \frac{\partial \rho}{\partial t} - \frac{\partial p}{\partial x_i} + \rho f_i - \frac{\partial}{\partial x_j}(\rho v_i v_j) - v_i \frac{\partial \rho}{\partial t}$$

$$\frac{\partial}{\partial t}(\rho v_i) + \left[\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}(\rho v_i v_j)\right] = \rho f_i$$

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$$\rightarrow \frac{\partial(\rho v_i)}{\partial t} + \frac{\partial}{\partial x^j}\left(\rho v_i v_j + P\delta_{ij}\right) = f_i f$$

Define

$$\boxed{\Pi_{ij} = P\delta_{ij} + \rho v_i v_j}$$

Momentum Flux Tensor

Then

$$\boxed{\frac{\partial(\rho v_i)}{\partial t} = -\frac{\partial}{\partial x^j}\Pi_{ij} + f_i f}$$

Euler equation in presence of external forces in Cartesian Tensor Notation.

Q22) We derived,

$$\frac{\partial \bar{B}}{\partial t} = \bar{\nabla} \times (\bar{v} \times B) + \eta \nabla^2 \bar{B}$$

$$\eta = \frac{1}{\mu\sigma}$$

Show that in ideal situation, ie $\eta = 0$

This leads to $\frac{d}{dt}\int_S \bar{B} \cdot ds = 0$ | Alfvén Theorem of flux freezing

This tells about the magnetic flux conservation across any arbitrary surface element S as S moves inside a fluid.

However, if $\eta \neq 0$ (Resistive MHD) there is no such conservation.

Am In ~~eta~~ non-relativistic regime, $\frac{v}{c} << 1$

And when $\frac{\partial \bar{E}}{\partial t} \simeq 0$.

we can write from Maxwell's equation in cgs units

* $\nabla \times \bar{B} = \frac{4\pi}{c} \vec{j} = \mu_0 \vec{j}$ (SI)

* $\frac{\partial \bar{B}}{\partial t} = -c (\nabla \times \bar{E})$. ..... (*1)

* $\nabla \cdot \bar{B} = 0$

Now, * $\vec{j} = \sigma \left(\bar{E} + \frac{\vec{v}}{c} \times \bar{B} \right)$

$\therefore \nabla \times \bar{B} = \frac{4\pi}{c} \sigma \left(\bar{E} + \frac{\vec{v}}{c} \times \bar{B} \right)$

$\frac{c}{4\pi\sigma} (\nabla \times \bar{B}) - \frac{1}{c} (\vec{v} \times \bar{B}) = \vec{E}$

Put in eq (*)

$\frac{\partial \bar{B}}{\partial t} =$

$-c \left[-\frac{1}{c} \nabla \times (\vec{v} \times \bar{B}) + \frac{c}{4\pi\sigma} \nabla \times (\nabla \times \bar{B}) \right]$

$\nabla \times (\vec{v} \times \bar{B}) - \frac{c^2}{4\pi\sigma} \nabla \times (\nabla \times \bar{B})$

$\nabla \times (\nabla \times \bar{B}) = \nabla (\nabla \cdot \bar{B}) - \nabla^2 B$.

$= 0$

$\boxed{\frac{\partial \bar{B}}{\partial t} = \nabla \times (v \times \bar{B}) + \eta \nabla^2 \bar{B}}$

$\eta = \frac{1}{\mu\sigma} = \frac{c^2}{4\pi\sigma}$.

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ARC

For $\eta = 0$ Pg-300, Physics of fluids, Plasma

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times (\vec{v} \times \vec{B}) \quad \cdots (1)$$

↳ The following theorem is true for any vector field $\vec{B}$.

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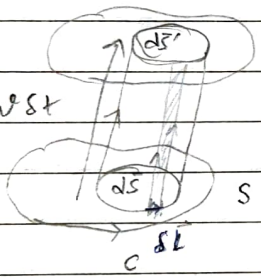
To proceed with proof, note that flux $\int_S \vec{B} \cdot d\vec{s}$ linked with surface S, can change with time due to 2 reasons →

① intrinsic variation in $\vec{B}$

② Motion of S.

∴ Mathematically,

$$\frac{d}{dt} \int_S \vec{B} \cdot d\vec{S} = \int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S} + \int_S \vec{B} \cdot \frac{d}{dt}(d\vec{S})$$



dS, dS' make up ends of cylinder

The vector area of a side strip of this cylinder is $(\vec{v}\,\delta t) \times \delta\vec{l}$

Since vector area $\oint d\vec{S} = 0$ for a closed ~~surface~~ surface

∴ Surfaces of ~~cy~~ cylinder satisfy

$$d\vec{S}' - d\vec{S} - \delta t \oint \vec{v} \times \delta\vec{l} = 0$$

If $\vec{B} = \vec{\omega}$ (Vorticity) this is called Kelvin's Theorem.

$\mathcal{R}_M = \frac{VB/L}{\lambda B/L^2} = \frac{LV}{\lambda}$

This is magnetic Reynolds Number

In laboratory $\mathcal{R}_M \ll 1$ $\frac{\partial \vec{B}}{\partial t} \approx \eta \nabla^2 \vec{B}$

In Astrophysics, $\mathcal{R}_M \gg 1$

$$\frac{\partial \vec{B}}{\partial t} \approx \nabla \times (\vec{v} \times \vec{B})$$

Line integral is taken around the surface element $d\vec{S}$.

It then follows that,

$$\frac{d}{dt}(d\vec{S}) = \lim_{\delta t \to 0} \frac{d\vec{S}' - d\vec{S}}{\delta t} = \oint \vec{v} \times \delta\vec{l}$$

$$\therefore \int_S \vec{B} \cdot \frac{d}{dt}(d\vec{S}) = \int \oint \vec{B} \cdot (\vec{v} \times \delta\vec{l}) = \int \oint (\vec{B} \times \vec{v}) \cdot \delta\vec{l}$$

① Line integral around each dS

② Sum up each such dS - integration over S. observe.

③ $\int \oint_S \equiv \oint_C$ line integral around C - curve.

∴ From Stokes theorem

$$= \oint_C (\vec{B} \times \vec{v}) \cdot \delta\vec{l} = \int_S \nabla \times (\vec{B} \times \vec{v}) \cdot d\vec{s}$$

$$\therefore \frac{d}{dt} \int_S \vec{B} \cdot d\vec{S} = \int_S \left[\frac{\partial \vec{B}}{\partial t} - \nabla \times (\vec{v} \times \vec{B}) \right] = 0 \,!!$$

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Note.

This type of model is useful in non-relativistic process where plasma motions are essentially constant.

i.e, time scale, $\tau \gg 1/\omega_p$ ω_p = plasma frequency.

Eg:- Stellar winds.

$$\frac{D\bar{v}}{Dt} = \frac{\partial \bar{v}}{\partial t} + (\bar{v}\cdot\nabla)\bar{v}$$

$$= -\frac{1}{\rho}\nabla P + \vec{f} + \frac{1}{\rho c}(\bar{J}\times\bar{B})$$

In MHD model:

$$\nabla \times \bar{B} = \frac{4\pi}{c}\bar{J}$$

$$\frac{\partial \bar{v}}{\partial t} + (\bar{v}\cdot\nabla)\bar{v} = -\frac{1}{\rho}\nabla P + \vec{f} + \frac{1}{4\pi\rho}(\nabla\times\bar{B})\times\bar{B}$$

Using identity.

$$\nabla(\bar{A}\cdot\bar{B}) = A\times(\nabla\times\bar{B}) + \bar{B}\times(\nabla\times\bar{A}) + (\bar{A}\cdot\nabla)\bar{B} + (\bar{B}\cdot\nabla)\bar{A}$$

For $A = B$, it is easy to see

$$(\nabla\times\bar{B})\times\bar{B} = (\bar{B}\cdot\nabla)\bar{B} - \nabla\left(\frac{B^2}{2}\right)$$

Euler eqn in magnetic field.

$$\frac{\partial \bar{v}}{\partial t} + (\bar{v}\cdot\nabla)\bar{v} = \vec{f} - \frac{1}{\rho}\nabla\left(P + \frac{B^2}{8\pi}\right) + \frac{(\bar{B}\cdot\nabla)\bar{B}}{4\pi\rho}$$

↓ external forcing

Magnetic pressure

related to tension along magnetic field lines, i.e B-field is associated with an isotropic pressure + tension along lines

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Q19) We calculated the collisional relaxation time for self-gravitating system.

See ARC, book, and express.

$$T_{relax} \text{ ~~T_{rel}~~} = 10^{10} \frac{v^3}{n}$$

↓

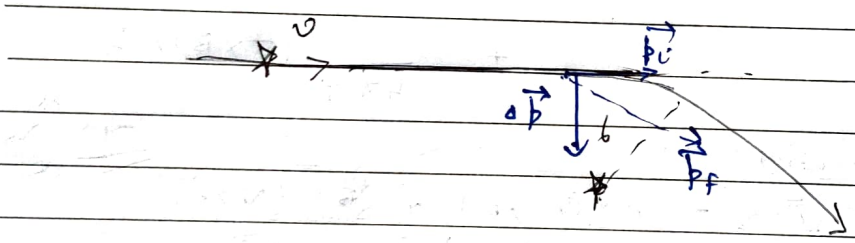
assume $m = M_\odot$.

Calc. $T_{rel} = ?$

Ans.

Virial Theorem is valid for any gravitationally bound stellar system in steady state, irrespective of whether the system is collisionally relaxed or not.

Using this we try to estimate the collisional relaxation time ~~for~~ of a stellar system.



consider

a galaxy / cluster ~~of~~ with stars of mass = m.

v = velocity of star.

We define a a collision — if the deflection of the trajectory of star ~~the~~ involves

$$|\Delta\vec{p}| \cong |\vec{p}_i| \quad \text{(initial mom).}$$

· consider a limiting collision scenario in above figure.

we need to find $b = ?$

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Read B&T cosmology Part

When two stars are close,

They will be for an interval: $\Delta t \approx \frac{b}{v}$

and ~~Gravitation~~ Gravitational force $\approx \frac{Gm^2}{b^2}$ upto 1st order

$\therefore$ $|\Delta p| \approx F_g \Delta t$

$$|\Delta p| \approx \frac{Gm^2}{b^2} \cdot \frac{b}{v} = \frac{Gm^2}{bv}$$

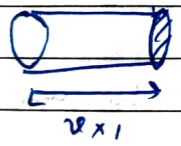
$$\vec{p}_i + \Delta \vec{p} = p_f$$

Since $|\Delta \vec{p}_i| \approx p_i$

$$\frac{Gm^2}{bv} = mv$$

$$b \approx \frac{Gm}{v^2}$$

Impact Parameter



In unit time moving star sweeps out a volume $\pi b^2 v$ within which another star has to lie for a collision to take place.

If n = No. density of stars

$\therefore$ $\frac{\text{No. of collision}}{\text{Time}} = \pi b^2 v n$

$\therefore$ We can define Relaxation Time (T_{rel}) — collision time in which the memory of any initial velocity is effectively erased.

$$T_{rel} = \frac{\text{Time}}{\#\text{collisions}} = \frac{1}{n \pi b^2 v} \approx \frac{v^3}{\pi n G^2 m^2}$$

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Put, $m \simeq M_\odot$

$v = ?$ Km/s

$n =$ No. / pc^3.

Then,

$$T_{rel} \approx 10^{10} \left(\frac{v}{Km/s}\right)^3 \left(\frac{pc^{-3}}{n}\right) \text{ yr}$$

For Galaxy	$v = 100$ Km/s, $n = 0.1\ pc^{-3}$	$T_{rel} = 10^{17}$ yr.
Open cluster	$v = 0.5$, $n = 1$	$T_{rel} = 10^9$ year
Globular Cluster	$v = 10$, $n = 10^3$	$T_{rel} = 10^{10}$ year.

Q15) Standard solar model provides the core density $\rho_{c,\odot}$ and temperature $T_{c,\odot}$ as

$\rho_{c,\odot} = 1.53 \times 10^5$ kg/m^3.

$T_{c,\odot} = 1.57 \times 10^7$ K.

Show electron degeneracy is not satisfied for the sun.

For sirius B,

$\rho_{c,B} = 3 \times 10^9$ kg/m^3.

$T_{c,B} = 7.6 \times 10^7$ K.

Show that electron degeneracy is achieved for Sirius B.

About White Dwarfs.

They are NOT White → they come in variety of colours. → 5×10^3 K → 80×10^3 K.

The largest group called DA- White Dwarf – display only pressure broadened by drogen absorption lines.

DB → He line

DC → continuum

DQ → Carbon lines

DZ – Metal lines

How was T_c calculated?

$$\frac{dT}{dr} = -\frac{3}{4ac}\frac{\bar{\kappa}\rho}{T^3}\frac{L_r}{4\pi r^2}$$

$$\frac{T_{wd} - T_c}{R_{wd} - 0} = -\frac{3}{4ac}\cdot\frac{\bar{\kappa}\rho}{T_c^3}\cdot\frac{L_{wd}}{4\pi R_{wd}^2}$$

T_{wd} = Surface Temp. of white Dwarf.

Let $\frac{T_{wd}}{T_c} << 1$

we. $\bar{\kappa} = 0.02\ m^2\ kg^{-1}$. (for electron scattering)

$$T_c \approx \left[\frac{3}{4ac}\bar{\kappa}\rho\frac{L_{wd}}{4\pi R_{wd}}\right]^{1/4}$$

$P_{rad} = \frac{1}{3}aT^4$, $a = \frac{4\sigma}{c}$

Stefan Boltzmann Law → $\rho \sim 3 \times 10^9\ kg/m^3$.

Astronomy → $(R_{wd} \sim 5.5 \times 10^6\ m.)$

$L_{wd} \sim 0.03 L_\odot$

orbit obst (mass)

We note, the condition of degeneracy is.

Claim

$$\frac{T}{\rho^{2/3}} < D. \qquad D \equiv \frac{1261}{\phi} K m^2 kg^{-2/3}$$

∴ The smaller the value for $T\rho^{-2/3}$, more degenerate is the gas.

For Sun

$$\frac{T}{\rho^{2/3}} = \frac{1.57 \times 10^7}{(1.53)^{2/3} (10^5)^{2/3}}$$

$$= 5488.31 \; K m^2 kg^{-2/3}.$$

∴ No

~~For Sirius B~~

The Sun will develop a degeneate helium core while on RGB of H-R diagram, leading eventually to a core helium flash. Later, on the asymptotic giant branch, the progenitor of a carbon-oxygen white dwarf will form in the core to be revealed when the sun's surface layers are ejected as a planetary nebula.

~~for~~ Sirius B;

$$\frac{T}{\rho^{2/3}} = 37 \; K m^2 kg^{-2/3} \quad << D$$

So complete degeneracy is a valid assumption for sirius B.

Proof

We have, fermi Energy

$$\varepsilon_F = \frac{\hbar^2}{2m} (3\pi^2 n)^{2/3}.$$

$n = \frac{N_e}{L^3}$ n is no. of electrons per unit volume

m = Mass of e^-.

For full ionization, the number density of electrons is —

$$n_e = \left(\frac{\#\ e^-}{\#\ \text{Nucleons}}\right)\left(\frac{\#\ \text{Nucleons}}{\text{Volume}}\right) = \left(\frac{Z}{A}\right)\frac{\rho}{m_H}$$

$m_H \cong m_p \cong m_n$

Z = Atomic Number

A = No. of Nucleons.

ρ = Density of matter

$$\therefore \varepsilon_F = \frac{\hbar^2}{2m}\left[3\pi^2\left(\frac{Z}{A}\right)\frac{\rho}{m_H}\right]^{2/3}$$

Now we ~~compare~~ see, average thermal energy of e^- $= \frac{3}{2} K_B T$.

∴ Roughly, if $\frac{3}{2} K_B T < \varepsilon_F$

then an average e^- will be unable to make a transition to an unoccupied state, and the e^- gas will be degenerate.

$$\therefore \frac{3}{2} K_B T < \frac{\hbar^2}{2m}\left[3\pi^2\left(\frac{Z}{A}\right)\frac{\rho}{m_H}\right]^{2/3}$$

$$\frac{T}{\rho^{2/3}} < \frac{\hbar^2}{3 m_H K_B}\left[\frac{3\pi^2}{m_H}\left(\frac{Z}{A}\right)\right]^{2/3} = 1261\ K\,m^2\,Kg^{-2/3} = D$$

Typically $\frac{Z}{A} \approx 0.5$ for most Nuclear Matter in stars.

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Q17) In liquid drop model of an atomic nucleus, a nucleus with mass number A has a radius of $r_0 A^{1/3}$.

$r_0 = 1.2 \times 10^{-15}$ m

Find the density of this nucleus model.

Compare this with Neutron star.

Density = $\frac{\text{Total Mass of Nucleon}}{\text{Total Volume}}$.

$$= \frac{(m_H \cdot A)}{\frac{4}{3}\pi r^3} = \frac{m_H A}{\frac{4}{3}\pi r_0^3 A}$$

$$\rho = \frac{3 m_H}{4\pi r_0^3}$$

$$= \frac{3 \times 1.67 \times 10^{-27}}{4\pi \times (1.2)^3 \times 10^{-45}} \text{ kg/m}^3$$

$$\approx \frac{3 \times 1.67}{12 \times 1.728} \times 10^{18}$$

$$\approx 2.3 \times 10^{17} \text{ kg/m}^3$$

For a Neutron star, $A \approx 10^{57}$;

NS is held together by gravity and supported by Neutron Degeneracy Pressure.

It can be shown

$$R_{ns} \approx \frac{(18\pi)^{2/3}}{10} \frac{\hbar^2}{G M_{ns}^{1/3}} \left(\frac{1}{m_H}\right)^{8/3}$$

For WD $M_{wd} V_{wd}$ = constant

For $M_{ns} = 1.4 M_\odot$,

We would get a value ~ 10 km = R_{ns}.

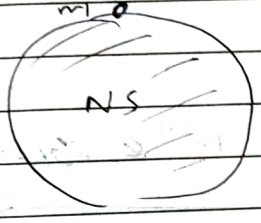
In that case, we would get an average density. $\rho = \frac{M_{ns}}{\frac{4}{3}\pi R_{ns}^3}$

Refer to Pg-597 Carroll & Ostlie. → $\rho \approx 6.65 \times 10^{17}$ kg/m^3

Density of Neutron star inner crust.

For outer crust $\rho \sim 4 \times 10^{14}$ kg/m^3

(18)



Assuming 1st Mass is on the surface

Flux freezing $\Phi = \int_S \vec{B} \cdot d\vec{A}$ is conserved.

$B_i 4\pi R_i^2 = B_f 4\pi R_f^2$.

Lower $F_1 = \frac{GMm}{R_{ns}^2}$

$h = 1$cm (Height)

Since, $\frac{h}{R_{ns}} \ll 1$.

Upper $F_2 = \frac{GMm}{(R_{ns}+r)^2} = \frac{GMm}{R_{ns}^2 \left(1+\frac{r}{R_{ns}}\right)^2}$

$\therefore F_2 = \frac{F_1}{\left(1+\frac{h}{R_{ns}}\right)^2}$

$\therefore \left(\frac{F_1}{F_2}\right)_{ns} = \left(1+\frac{h}{R_{ns}}\right)^2$

$\simeq 1 + \frac{2h}{R_{ns}} = 1 + \frac{2 \times 0.01}{10 \times 10^3}$

$= 1 + 2 \times 10^{-6}$

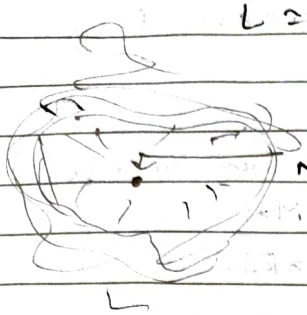
For Earth,

$\left(\frac{F_1}{F_2}\right)_\oplus = \left(1+\frac{2h}{R_\oplus}\right) = 1 + \frac{2 \times 0.01}{6400 \times 10^3} 10^{-9} = 1 + 3.12 \times 10^{-9}$

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(16) Crab Nebula

$L \simeq 10^{31} W$ → by radiation away of magnetic dipole energy.



$m_{NS} = 1 M_\odot$

$R_{ns} = 10 Km.$

It is observationally found that the pulsar has pulse timeperiod $= 0.03 s$.

What should be the slowdown Rate in pulse time period?

Observed Slowdown Rate $= 10^{-13}$ s/s.

Why will it slow down?

① Energy lost by being radiated away by Magnetic Field.

Assuming $v = 0$ is i.e Pulsar is stationary in space.

It's Energy,

$$E = \frac{1}{2} I_p \omega^2 - \frac{3}{5} \frac{G M_p}{R_p^2}$$

KE, ang. vel., Pot. En.

$I_p = \frac{2}{5} M_p R^2$ (Rigid sphere)

$\omega = \frac{2\pi}{P}$ P = Period

$\dot{P} = ?$

Now,

Mass, Radius is constant.

∴

∴ Rate of Energy Loss

$$E = \frac{1}{2} I_p \cdot \frac{4\pi^2}{P^2}$$

$$E = \frac{2\pi^2 I_p}{P^2}$$

$$\frac{dE}{dt} = -\frac{4\pi^2 I_p}{P^3} \dot{P} \qquad P = 0.033$$

$$= 10^{31} \text{ W (given)}$$

Synchrotron Radiation.

Easy to calc. $\dot{P}$

ie accelerating expansion of Nebula.

$$\dot{P} = -\frac{10^{31} \times P^3}{4\pi^2 I_p}$$

$$I_p = \frac{2}{5} M_p R_p^2$$

$$= \frac{2}{5} \times M_\odot \times (10^8)$$

$$\dot{P} = -\frac{10^{31} \times 27 \times 10^{-6} \times 5 \times 10^{-14}}{4\pi^2 \times 2 \times 10^{30} \times 10^8 \times 2}$$

$$\simeq \left(\frac{27 \times 5}{4\pi^2 \times 8}\right) 10^{-14}$$

Cal.

Period of slowing down.

$$\dot{P} = 9.37 \times 10^{-14} \text{ s/s}$$

$$\simeq 10^{-13} \text{ s/s}$$

Close to the value.

(13) The Q value of a reaction is the amount of energy released during the reaction.

Calculate the Q value for each step of pp1 chain.

Masses of $_1^1H$ = 1.0078 amu.

$_1^2H$ = 2.0141 amu.

$_2^3He$ = 3.0160 amu

$_2^4He$ = 4.0026 amu.

Neglect all small particle masses.

In generals a p-p chain reaction can occur only if T (or K. Energy) is high enough to overcome their mutual electrostatic / Coulomb repulsion.

Step1:

$_1H^1 + {}_1H^1$

$\tau_{1/2} \sim 10^7$ years.

slow af.

Step2:

$_1H^1 + (_1D^2)$ $(+ e^+ + \nu_e + \gamma)$

$\tau_{1/2} \sim 4s$

fast af.

$(_2He^3) + \gamma \rightarrow$ stays for abt 400 years

From Here there are 4 possible

paths to produce He^4.

Branches
- pp1 → 86% probability
- pp2 → 14% ← $^1H_1 + {}^1H_1$
- pp3 → 0.11%
- ppn → বাকিটা

Step3:

pp I Branch (T ~ 10–14 MK)

$$_2He^3 + {}_2He^3 \longrightarrow {}_2He^4 + 2\,{}_1H^1$$

pp II (T ~ 14–23 MK)

$$_2He^3 + {}_2He^4 \longrightarrow {}_4Be^7 + \gamma$$

$$_4Be^7 + e^- \longrightarrow {}_3Li^7$$

$$_3Li^7 + {}_1H^1 \longrightarrow 2\,{}_2He^4$$

pp III (T > 23 MK)

$$_2He^3 + {}_2He^4 \longrightarrow {}_4Be^7 + \gamma$$

$$_4Be^7 + {}_1H^1 \longrightarrow {}_5B^8$$

$$_5B^8 \longrightarrow {}_4Be^8 + e^+ + \nu_e$$

$$_4Be^8 \longrightarrow 2\,{}_2He^4$$

PP IV (Rare)

$$_2He^3 + {}_1H^1 \longrightarrow {}_2He^4 + e^+ + \nu_e$$

For $M_* < 1.3 M_\odot$ → pp chain dominant

$M_* > 1.3 M_\odot$ → CNO Cycle.

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Now

① $_1H^1 + {}_1H^1 \longrightarrow {}_1D^2 + {}_1e^{+0}$

$Q_1 = ?$

For a general Reaction

$A + B \longrightarrow X + Y$

$Q = [m_A + m_B - (m_X + m_Y)] \times 931.5 \frac{MeV}{c^2}$

Δm.

We know

$m_n \simeq 939.57 \ MeV/c^2$

$m_p \simeq 938.27 \ MeV/c^2$

$m_e \sim 0.511 \ MeV/c^2$

$\Delta m = (2 \times 1.0078 - 2.0141) = 0.0015$

$\therefore Q_1 = 1.397 \ MeV$

② $_1H^1 + {}_1D^2 \longrightarrow {}_2He^3$

$\Delta m = (2.0141 + 1.0078 - 3.0160)$

$= 0.0059$

$Q_2 = 5.4985 \ MeV$

③ $2\ {}_2He^3 \longrightarrow {}_2He^4 + 2\ {}_1H^1$

$\Delta m = (2 \times 3.0160 - 4.0026 - 2 \times 1.0078)$

$= 12.8547 \ MeV$

Total energy released in p-p cycle

$Q = Q_1 + Q_2 + Q_3 = 2 \times 1.397 + 2 \times 5.495 + 12.854$

$= 26.638$

$\simeq 26.64$ MeV

$2\,{}_1H^1 + 2\,{}_1H^1 \rightarrow {}_1D^2$

$2\,{}_1H^1 + 2\,{}_1D^2 \rightarrow 2\,{}_2He^3$

$2\,{}_2He^3 \rightarrow {}_2He^4 + 2\,{}_1H^1$

$4\,{}_1H^1 \rightarrow {}_2He^4 + 2\,{}_1e^{0+}$

Fraction of Mass converted into Energy

$$4\,{}_1H^1 \longrightarrow 2\,He^4 + 2\,{}_1e^+$$

Neglect.

Initial Mass of 4 proton = 4×1.0078
$= 4.0312$ amu

Final Helium mass = 4.0026.

$\Delta m = 4.0312 - 4.0026$
$= 0.0286$

This much released the energy.

$\therefore$ Fraction $= \frac{1}{4} \times \frac{0.0286}{4.0312} = 0.1773\ \%$

$\frac{1}{4}$ 4 Hydrogen

Q.14) one can make rough estimate for main seq. lifetime T_{MS}.

Assuming sun will move off main sequence when the ~~centre~~ central 10% of hydrogen will get converted into helium, estimate T_{MS}.

Current luminosity of sun = 3.86×10^{26} W $= L_\odot$

$T_{MS} \approx$ Total Energy generated in M.S / Luminosity

Lifetime

Mass of sun = 2×10^{30} kg

Assuming sun consists of entirely Hydrogen to start with

$$n_H = \left(\frac{2 \times 10^{30}}{m_H}\right) X = \left(\frac{M_\odot}{m_H}\right) X$$

$X = 0.7$ Hydrogen mass fraction

When 10% used

Total Energy generated $= Mc^2$

$$= \frac{1}{10} \times \left(\frac{M_\odot}{m_H}\right) \times m_H \times \frac{0.1773}{100} \times 9 \times 10^{16}$$

$$= \frac{2 \times 10^{30} \times 0.1773 \times 9 \times 10^{13} \times 0.7}{1.6} = 1.396 \times 10^{43}$$

$$\therefore T_{ms} = \frac{2 \times 10^{30} \times 0.1773 \times 9 \times 10^{13} \times 0.7}{3.86 \times 10^{26} \times 365 \times 86400} \text{ yr}$$

$$= \frac{1.39 \times 10^{43}}{3.86 \times 10^{26} \times 365 \times 86400}$$

$$= 1.147 \times 10^9 \text{ by}$$

Now

$$T_{ms} \propto \frac{1}{L}$$

$$\therefore \frac{T_{ms,\odot}}{T_{ms,L}} = \frac{L_L}{L_\odot} = \left(\frac{M_L}{M_\odot}\right)^{3.5}$$

lower end

$$T_{ms,L} = 1.147 \text{ by} \times (10)^{3.5} = 36.27 \text{ by}$$

Similarly upper end

$$T_{ms,U} = (1.147) \times (1/85)^{3.5} = 2.025 \times 10^{-7} \text{ by}$$