# Oscillations

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1. **Simple Harmonic Motion**
Begin with,
$$x(t) = A\sin(\omega t + \phi)$$
or
$$x(t) = c_1 e^{i\omega_o t} + c_2 e^{-i\omega_o t}$$
At normal mode, all masses will oscillate with the same frequency. N masses $\implies$ N normal modes.

- $(k, m_1, k, m_2, \dots m_N, k) \to X$
Consider the above configuration. Then **transverse** oscillations will perpendicular to x-axis $(y/z)$. while **longitudinal** oscillations will be along the line. We can safely assume the displacement of each mass as:
$$\ddot{x_n} = -\omega^2 x_n \ \forall\ n$$

- $(k_1, m, k_2, m, k_1)$
Here, we have one in-phase and other out of phase modes. They are:
$$\omega_1 = \sqrt{\frac{k_1}{m}}\ ;\ \omega_2 = \sqrt{\frac{k_1 + 2k_2}{m}}$$
When, the normal modes are superimposed it results in the formation of *beads*.
$$T_b = \frac{2\pi}{\omega_2 - \omega_1}$$

- **Beaded String:**
$(k, m_1, \dots k, m_{n-1}, k, m_n, k, m_{n+1}, \dots, m_N, k)$
Note that: all the masses(=m) are identical separated by a distance $a$, the indexes represent their positions. The total length of string is $= a(N+1) = L$ For **transverse** oscillations,
$$\ddot{y_n} = -\omega_o^2[2y_n - y_{n-1} - y_{n+1}]$$
where, $\omega_o^2 = T/ma$. Now, substitute
$$y_n = A_n \cos\omega_{\mathcal{M}} t\ ;\ A_p = C\sin n\theta$$
then use the boundary values $A_0 = 0$ and $A_{N+1} = 0$ if fixed at *both ends*, while $|A_N| \to$ max ,if *free at one* end. For the former case, one gets
$$A_n^{\mathcal{M}} = C_{\mathcal{M}} \sin\left(\frac{\mathcal{M}}{N+1} n\pi\right)$$
$$\omega_{\mathcal{M}} = 2\omega_o \sin\left(\frac{\pi}{N+1}\frac{\mathcal{M}}{2}\right)$$
And, Dispersion relation $\boxed{\omega_{\mathcal{M}} = 2\omega_o \sin\left(\frac{1}{2}\kappa a\right)}$

- **Continuum Limit**: $na \simeq x$
Wave number, $\kappa = \frac{\mathcal{M}\pi}{L} = \frac{2\pi}{\lambda}$. Here, Dispersion relation $\boxed{\kappa = \frac{\omega_{\mathcal{M}}}{v}}$ (Standing waves)

- **Wave Equation**
$$\Box\psi = 0 \implies \frac{\partial^2\psi}{\partial t^2} = v^2\frac{\partial^2\psi}{\partial x^2}$$
  - Standing waves: $\psi = cf(x)g(t)$
  - Travelling waves: $\psi = \psi(x \pm vt)$

- **Velocities:**
$$v_G = \frac{\omega(\kappa)}{\kappa}\ \&\ v_P = \frac{\mathrm{d}\omega}{\mathrm{d}\kappa}$$

2. **Wavepacket**
Given a dispersion relation and amplitude $\mathrm{A}_{(\kappa)}$, we can get the equation of wavepackets as-
$$y = \int_{\kappa_1}^{\kappa_2} A_{(\kappa)} \exp(i(\kappa x - \omega t))\mathrm{d}\kappa$$

3. **Damped Harmonic Oscillator**
We have,
$$\ddot{x} + \alpha\dot{x} + \omega_o^2 x = 0$$
where, $\omega_o^2 = k/m$ and $\alpha = \gamma/m$. We assume a solution $x = Ae^{i\omega t}$, then
$$\omega = \frac{i\alpha}{2} \pm \sqrt{\omega_o^2 - \alpha^2/4} = \frac{i\alpha}{2} \pm \omega_1$$

**Case 1** $\omega_o^2 - \alpha^2/4 > 0$ Underdamped
$$x(t) = e^{-\alpha t/2}[A\sin\omega_1 t + B\cos\omega_1 t]$$

**Case 2** $\omega_o^2 - \alpha^2/4 < 0$ Overdamped
$$x(t) = e^{-\alpha t/2}[A\exp(\omega_1 t) + B\exp(-\omega_1 t)]$$

**Case 3** $\omega_o^2 - \alpha^2/4 = 0$ Critically damped
$$x(t) = e^{-\alpha t/2}[A + Bt]$$

4. **Forced Damped Oscillator**
$$\ddot{x} + \alpha\dot{x} + \omega_o^2 x = \frac{F_o}{m}\cos\omega t$$
The complementary solution, gives us the transient motion, while particular solution gives us the steady state equation (frequency of oscillation = forcing frequency). Solving similarly,
$$x = \frac{F_o/m}{\sqrt{(\omega_o^2 - \omega^2)^2 + \alpha^2\omega^2}}\cos(\omega t - \phi)\ ;\ \tan\phi = \left(\frac{\alpha\omega}{\omega_o^2 - \omega^2}\right)$$
$$\implies x(t) = A_{dis}\cos\omega t + A_{ab}\sin\omega t$$

5. **Work and Power**

$$\langle P\rangle_{in} = \frac{1}{T}\int_0^T F(t)\dot{x}\mathrm{d}t$$

$$\langle P\rangle_{dis} = \frac{1}{T}\int_0^T m\alpha\dot{x}^2\mathrm{d}t$$

Note: To maintain steady state oscillations,

$$\langle P\rangle_{in} = \langle P\rangle_{dis}$$

For $F(t) = F_o\cos\omega t, \langle P\rangle = F_o\omega A_{ab}/2$

$$= \frac{F_o^2}{2m\alpha}\left[\frac{1}{1+(\frac{\omega_o^2-\omega^2}{\omega\alpha})^2}\right]$$

Quality Factor, $Q = \frac{\omega_o}{\alpha}$. See Fig 2

6. **Damped driven coupled pendula**
Consider two pendula of identical bob($m$), and connected with a spring ($k_s$), the first one is driven by an harmonic oscillator. This is an example of *band pass filter*, Given $k_s$ small enough, it will allow only a certain band of frequencies to pass to other end. The equations are:

$$\ddot{x_1} + \alpha\dot{x_1} + \left(\frac{g}{L}+\frac{k_s}{m}\right)x_1 - \frac{k_s}{m}x_2 = \frac{F(t)}{m}$$

$$\ddot{x_2} + \alpha\dot{x_2} + \left(\frac{g}{L}+\frac{k_s}{m}\right)x_2 - \frac{k_s}{m}x_2 = 0$$

7. **Infinite coupled pendula system**
See Fig 3. Note: $K$- spring constant. $k = \frac{2\pi}{\lambda}$ is wave number and constant. $\kappa = \frac{1}{\delta}$ is Attenuation per unit distance. $\omega_o^2 = g/l$.

(a) Discrete Case

$$\ddot{\psi_n} = -\omega_o^2\psi_n + \frac{K}{M}[2\psi_n - \psi_{n+1} - \psi_{n-1}]$$

Here, $\psi_n = A_n\cos(\omega t+\phi)$ ; $A_n = C\sin nka$.
$\omega_o$ is Low Cutoff Frequency.

Dispersion relation $\boxed{\omega^2 = \omega_o^2 + \frac{4K}{M}\sin^2 ka/2}$

Dispersive range (*Pass band*): $\omega_o < \omega < \sqrt{\omega_o^2 + \frac{4K}{M}}$.

(b) Continuos Case
It is a high pass filter. **Klein Gordon Wave Equaion:**

$$\frac{\partial^2\psi}{\partial t^2} = -\omega_o^2\psi + \frac{Ka^2}{M}\nabla^2\psi$$

Here, $\psi = f(x)\cos(\omega t+\phi)$

**Case-1** $\omega^2 < \omega_o^2$
This medium is known as reactive, as it supports *exponentially decaying* waves (without energy dissipation!).

$$f(x) = c_1 e^{-\kappa x}$$

Dispersion relation $\boxed{\omega^2 = \omega_o^2 - \left(\frac{Ka^2}{M}\right)\kappa^2}$

**Case-2** $\omega^2 > \omega_o^2$
This medium is known as dispersive, as it supports *sinusoidal* waves, (acting transparent).

$$f(x) = c_1\sin kx + c_2\cos kx$$

Dispersion relation $\boxed{\omega^2 = \omega_o^2 + \left(\frac{Ka^2}{M}\right)k^2}$

8. **The ionosphere**
The Earth's ionosphere (driven by radio stations) is dispersive to frequencies above $\nu_p = \omega_p/2\pi$, plasma oscillation frequency. It's dispersion relations are-

$$\omega^2 = \omega_o^2 - c^2\kappa^2 \;\forall\; \omega < \omega_p$$
$$\omega^2 = \omega_o^2 + c^2k^2 \;\forall\; \omega > \omega_p$$

So, when the ionosphere is reactive, it does not absorb any energy but instead reflects back the radio wave (AM) over a broader region. At FM, TV frequencies the ionosphere is transparent.

Now, observe Fig 4. To understand this we relate it analogous to coupled pendula system. Consider

$$\omega_o^2 = \frac{g}{l_1} < \omega^2 \qquad \forall\; 0 < z < L \;(1)$$
$$\omega_o^2 = \frac{g}{l_2} > \omega^2 \qquad \forall\; z > L \;(2)$$

Region 2 is called plasma, where the medium becomes reactive and wave is reflected back.

9. **Reflection**
Imagine a rope being driven up and down by a force $F$ at one end. The tension in the rope is $T_o$ and linear mass density is $\mu$. The wave created as a result is propagating along $+x$ direction.

$$\therefore F \simeq T_o\frac{\partial y}{\partial x} = -Z\frac{\partial y}{\partial t}$$

where $Z = \sqrt{T_o\mu}$ is called the **impedance** of the rope (medium). Consequently transmitted power can be given by $\mathcal{P} = Z(\frac{\partial y}{\partial t})^2$. Now we formulate the equations of transmitted and reflected wave, travelling from $Z_1 \to Z_2$. At the interface-

$$y_{in} + y_{ref} = y_{tr} \qquad (1)$$
$$-Z_1\frac{\partial y_{in}}{\partial t} + Z_1\frac{\partial y_{ref}}{\partial t} = -Z_2\frac{\partial y_{tr}}{\partial t} \qquad (2)$$

Solving them we obtain,

$$y_{ref} = R_{12}\; y_{in}$$
$$y_{tr} = T_{12}\; y_{in}$$
$$T_{12} = 1 + R_{12}$$

$$\boxed{R_{12} = \frac{Z_1 - Z_2}{Z_1 + Z_2} = -R_{21}}$$

$$\boxed{T_{12} = \frac{2Z_1}{Z_1 + Z_2}}$$

## Important Diagrams

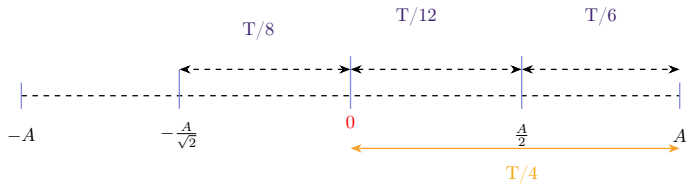



Figure 1: For a particle undergoing Linear SHM.

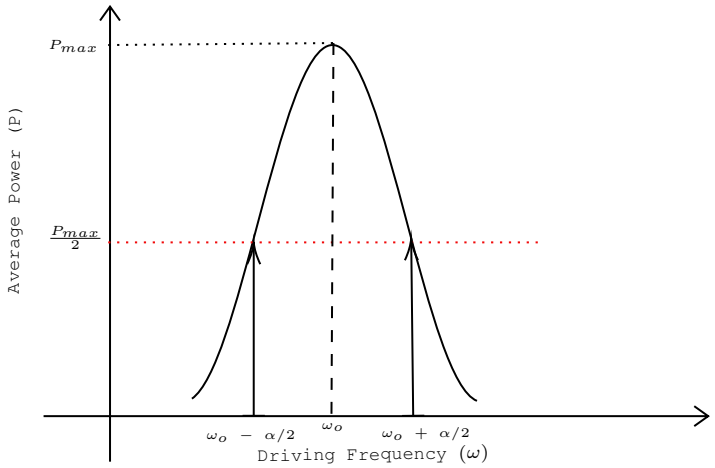



Figure 2: Plot showing the power dissipated during with the variation of driving frequency. The red line represents the FWHM

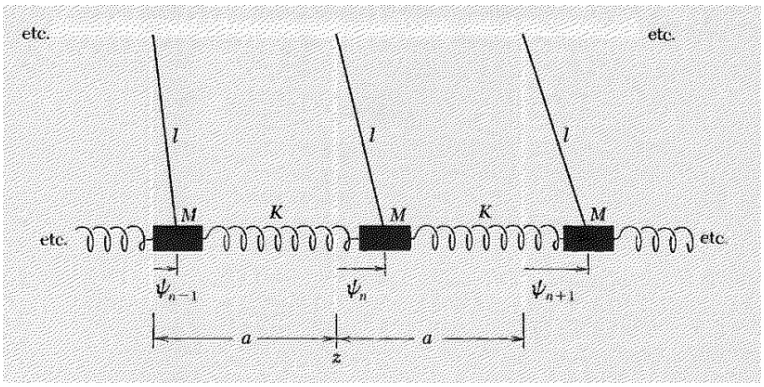



Figure 3: Coupled Pendulums with unspecified boundary conditions.

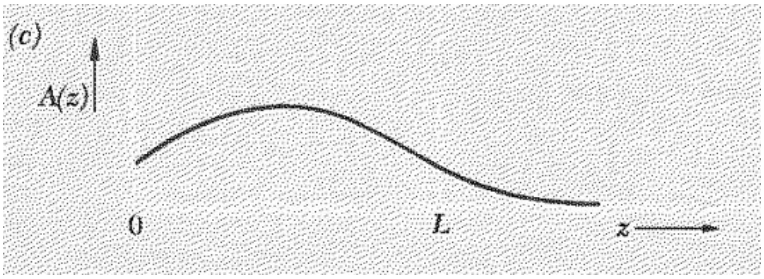



Figure 4: Wave Penetrance into reactive region of atmosphere, near the lowest resonance frequency of system.