ELECTROSTATICS

by Susnata Chattopadhyay[22MS077], IISER-K

1. **Useful Formulas**

$$\int_{P_1}^{P_2} \boldsymbol{\nabla}\phi \cdot d\mathbf{r} = \phi(P_2) - \phi(P_1)$$

over any curve C

$$\oint_S \mathbf{E} \cdot \mathbf{dS} = \int_\tau \boldsymbol{\nabla} \cdot \mathbf{E}\ d\tau$$

where τ is the volume, enclosed by a closed surface S

$$\oint_C \mathbf{E} \cdot \mathbf{dr} = \int_S (\boldsymbol{\nabla} \times \mathbf{E}) \cdot \mathbf{dS}$$

where S is the surface bounded by curve C

2. **Coulomb's Law and Electric Field**

$$\vec{F} = \frac{1}{4\pi\epsilon_o}.q\sum_i \frac{\dot{q}_i(\vec{\mathbf{r}} - \vec{\mathbf{r_i}}')}{|\vec{\mathbf{r}} - \vec{\mathbf{r_i}}'|^3} = q\vec{E}$$

Here $\vec{\mathbf{r}}-$ position vector of point, where it is calculated. $\vec{\mathbf{r}}'-$ position vector of source wrt origin.

For continuous charges-

$$\vec{E} = \frac{1}{4\pi\epsilon_o}\int \frac{dq(\vec{\mathbf{r}} - \vec{\mathbf{r}}')}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|^3}$$

where

$$dq = \lambda(\vec{\mathbf{r}}')dx'$$
$$dq = \sigma(\vec{\mathbf{r}}')dA'$$
$$dq = \rho(\vec{\mathbf{r}}')d\tau'$$

3. **Maxwell's Equations(Statics)**

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho}{\epsilon_o} \iff \oint_S \mathbf{E} \cdot \mathbf{dS} = \frac{Q_{enc}}{\epsilon_o}$$

$$\boldsymbol{\nabla} \times \mathbf{E} = 0 \iff \oint_C \mathbf{E} \cdot \mathbf{dl} = 0$$

4. **Electric Potential**

$$V = \frac{1}{4\pi\epsilon_o}\int \frac{dq}{|\mathbf{r} - \mathbf{r}'|}$$

$$\mathbf{E} = -\boldsymbol{\nabla}V$$
$$\nabla^2 V = -\frac{\rho}{\epsilon_o}$$

and in charge free region:

$$\nabla^2 V = 0$$

Potential is defined as,

$$V_B - V_A = -\int_A^B \mathbf{E} \cdot \mathbf{dl}$$

Now, in Electrostatic conditions $\mathbf{E}$ is a conservative field. So it follows the 2nd maxwell's equation.

5. **Electrostatic Potential Energy**

$$U = \sum_i q_i V_i \Rightarrow U = \frac{1}{2}\int_\tau \rho(\mathbf{r}')V(\mathbf{r}')\ d\tau'$$

But, applying Gauss's Law, we obtain-

$$U = \frac{\epsilon_o}{2}\left(\oint_S \mathbf{E}V \cdot \mathbf{dS} + \int_\tau E^2 d\tau\right)$$

Hence, the total Energy in the field is,

$$U = \frac{\epsilon_o}{2}\int_{all\ space} E^2 d\tau$$

Energy Density is $\frac{\mathrm{d}U}{\mathrm{d}\tau} = \frac{1}{2}\epsilon_o E^2$

6. **Multipole Expansion**
$f(z)$ is a complex function, where $z \in \mathbb{C}$

- Monopoles: Any charge configuration, whose potential at large distances varies as:

$$V \sim \frac{1}{r}$$

Monopole Moment- $\mathcal{Q} = \sum q_i$
Field: $f(z) = z^{-1}$

- Dipoles: From equivalent definition,

$$V \sim \frac{1}{r^2}$$

Dipole Moment- $\vec{\mathbf{p}} = \sum q_i \vec{\boldsymbol{r_i}} \Rightarrow \int \vec{\boldsymbol{r}} dq$
Potential- $V = k\frac{\vec{\mathbf{p}} \cdot \hat{\boldsymbol{r}}}{r^2}$
Electric Field- $\frac{kp}{r^3}(2\cos\theta\ \hat{\boldsymbol{r}} + \sin\theta\ \hat{\boldsymbol{\theta}}) \iff f(z) = z^{-2}$

- Quadrupoles:

$$V \sim \frac{1}{r^3}$$

Field: $f(z) = z^{-3}$

Quadrupole moment is defined as:

$$\overleftrightarrow{Q} = \begin{bmatrix} Q_{xx} & Q_{xy} & Q_{xz} \\ Q_{yx} & Q_{yy} & Q_{yz} \\ Q_{zx} & Q_{zy} & Q_{zz} \end{bmatrix}$$

where,

$$Q_{ij} = \frac{1}{2}\sum_k q_k[3r_{k_i}r_{k_j} - r_k^2\delta_{ij}]$$

Here, $\delta_{ij} =$ `Kronecker delta` function
r_{k_i} = component of $\vec{\boldsymbol{r_k}}$ (k^{th} charge) along i axis.
$i \in \{x, y, z\}$

- Octupoles: $f(z) = z^{-4}$