Name - SUSNATA Chattopadhyay

Roll: 22MS077

PH3203 Assignment-5

Q) Show that under action of $\hat{U}_t \equiv e^{-i\hat{H}t/\hbar}$, any $|\psi\rangle \in H$

(i) $H = \lambda \vec{S}_1 \cdot \vec{S}_2$ → preserves general symmetry

(ii) $H = \lambda \vec{S}_1 \cdot \vec{S}_2 + \vec{B} \cdot (\vec{S}_1 + \vec{S}_2)$ ~~preserves~~ has only Rot. symmetry about B-axis.

Ans We assume the two interacting spins, are each of spin, $s_i = 1/2$

$\therefore m_s = \frac{1}{2}$ or $\frac{-1}{2} \leftarrow |\downarrow\rangle$

(S_z) ↓

$|\uparrow\rangle$

$\therefore$ S_z: -1, 0, 1

$S \equiv 0, 1$

$S_z = 0$

Let $|S, m\rangle$ be basis of the interacting joint system. $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$.

$\therefore$ we can write the basis of this joint system in terms of direct product basis states of individual spins.

Tensor product basis states

$$|1,1\rangle = |\uparrow,\uparrow\rangle \equiv |\uparrow\rangle_1 \otimes |\uparrow\rangle_2$$
$$|1,0\rangle = \frac{1}{\sqrt{2}}(|\uparrow,\downarrow\rangle + |\downarrow,\uparrow\rangle)$$
$$|1,-1\rangle = |\downarrow,\downarrow\rangle$$

} Symmetric.

$$|0,0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$$

} Antisymmetric Symmetry

(1)

Preserving general symmetry means, under action of $\hat{U}_t$, ~~any state~~ a symmetric $|\psi\rangle$ ($|\psi\rangle_{sym}$) remains symmetric, while antisymmetric state. $|\psi\rangle_{asym}$ remains antisymmetric!

∴ we have

$$H = A\,\bar{S}_1 \cdot \bar{S}_2 = \frac{A}{2}\left[S_{tot}^2 - S_1^2 - S_2^2 \right]$$

$$= \frac{A}{2}\left[S_{tot}^2 - \frac{3\hbar^2}{2} \right].$$

$$S_1^2 |m_1, s_1, m_2, s_2\rangle = s_1(s_1+1)\hbar^2 |m\rangle$$

$$\frac{1}{2}\left(\frac{3}{2}\right)$$

we know,

$$S^z_{tot} = S_1^z + S_2^z .$$

Check $[S^z_{tot}, S^2_{tot}] = 0$.

So, this means eigenstates of H, can be written in terms of quantum numbers 's' corresponding to S^2_{tot} and m corresponding to S^z_{tot}.

$$S^2_{tot} |s, m\rangle = \hbar^2 s(s+1) |s, m\rangle .$$

$s = 0$

$$S^2_{tot} |0, 0\rangle = 0$$

$s = 1$

$$S^2_{tot} |1, \cdot\rangle = 2\hbar^2 |1, \cdot\rangle .$$

$-1, 0, 1$

Then

$$\hat{H} |1,1\rangle = \frac{A\hbar^2}{4} |1,1\rangle$$

$$\hat{H} |1,0\rangle = \frac{A\hbar^2}{4} |1,0\rangle$$

$$\hat{H} |1,-1\rangle = \frac{A\hbar^2}{4} |1,-1\rangle$$

$$\hat{H} |0,0\rangle = -\frac{3\hbar^2}{4} A |0,0\rangle .$$

Clearly, we saw, each symmetric state is then a linear combination of $\{|1,1\rangle, |1,-1\rangle, |1,0\rangle\}$ while any antisymmetric state is just a ~~line~~ scalar multiple of $\{|0,0\rangle\}$.

So,

$$e^{-\frac{i}{\hbar}\hat{H}t}|1,1\rangle = e^{-\frac{i}{\hbar}\left(\frac{1}{4}\hbar^2\right)t}|1,1\rangle \quad (2)$$

$$\vdots$$

$$e^{-\frac{i}{\hbar}\hat{H}t}|0,0\rangle = e^{+\frac{i}{\hbar}\left(\frac{3}{4}\hbar^2\right)t}|0,0\rangle.$$

(C-1)

Let $|\Psi\rangle_{sym}$ denote a general symmetric state.

Then. $\{\alpha_i\} \in \mathbb{C}$

$$\hat{U}_t|\Psi\rangle_{sym} = e^{-\frac{i\hat{H}}{\hbar}t}\left(\alpha_1|1,1\rangle + \alpha_2|1,0\rangle + \alpha_3|1,-1\rangle\right)$$

$$= e^{-\frac{i\lambda\hbar t}{4}}\left(\alpha_1|1,1\rangle + \alpha_2|1,0\rangle + \alpha_3|1,-1\rangle\right)$$

$$= \beta\,|\Psi\rangle_{sym} \quad \in \mathcal{H}_{sym}.$$

Since $\beta \in \mathbb{C}$ is a scalar

Thus under time evolution. $|\Psi\rangle_{sym}$ remains symmetric

(C-2)

Consider $|\Psi\rangle_{asym} = \alpha\,|0,0\rangle$ $(\alpha \in \mathbb{C})$ be a antisymmetric state.

$$\therefore U_t|\Psi\rangle_{asym} = \alpha\, e^{i\left(\frac{3}{4}\hbar\right)t}|0,0\rangle.$$

$$= \gamma\,|0,0\rangle, \quad \gamma \in \mathbb{C}$$

Again Antisymmetric

D.

(ii) $\hat{H} = \underbrace{\lambda \bar{S}_1 \cdot \bar{S}_2}_{\hat{H}_{rot}} + \underbrace{\bar{B} \cdot (\bar{S}_1 + \bar{S}_2)}_{\hat{H}'} = \hat{H}_{rot} + \hat{H}'$

Now, consider

$[\hat{H}_{rot}, \bar{S}_{tot}] = [\lambda \bar{S}_1 \cdot \bar{S}_2, \bar{S}_1 + \bar{S}_2]$

$= \lambda \left([S_1^x S_2^x + S_1^y S_2^y + S_1^z S_2^z, S_1^x + S_2^x] \hat{i} + [,] \hat{j} + [,] \hat{k} \right)$

$[S_1^x S_2^x + S_1^y S_2^y + S_1^z S_2^z, S_1^x + S_2^x]$

$= [S_1^y S_2^y + S_1^z S_2^z, S_1^x] + [S_1^y S_2^y + S_1^z S_2^z, S_2^x]$

$= i\hbar \left(S_1^y S_2^z - S_1^z S_2^y + S_1^z S_2^y - S_1^y S_2^z \right) = 0!$

$\therefore$

And, similarly

$[\bar{S}_1 \cdot \bar{S}_2, S_1^y + S_2^y] = [\bar{S}_1 \cdot \bar{S}_2, S_1^z + S_2^z] = 0.$

Thus, $\boxed{[H_{rot}, \vec{S}_{tot}] = \vec{0}}$

$\Rightarrow (n_x \hat{i} + n_y \hat{j} + n_z \hat{k}) \cdot \left([H_{rot}, S^x_{tot}] \hat{i} + [H_{rot}, S^y_{tot}] \hat{j} + [H_{rot}, S^z_{tot}] \hat{k} \right) = 0$

for some, ~~an~~ arbitrary unit vector $\hat{n} = n_x \hat{i} + n_y \hat{j} + n_z \hat{k}$

$\Rightarrow$ ~~if~~ $n_x [H_{rot}, S^x_{tot}] + n_y [H_{rot}, S^y_{tot}] + n_z [H_{rot}, S^z_{tot}]$

$\Rightarrow \boxed{[H_{rot}, \hat{n} \cdot \vec{S}_{tot}] = 0}$

Hence, H_{rot} is rotationally symmetric about arbitrary unit vector $\vec{n}$.

$\therefore [H_{rot}, \bar{n} \cdot \bar{S}] = 0 \Rightarrow \hat{U}_\theta^\dagger H_{rot} \hat{U}_\theta = H_{rot}$; $\hat{U}_\theta = e^{-\frac{i}{\hbar} \theta \bar{n} \cdot \vec{S}}$

Now, let magnetic field, $\bar{B} = B\hat{e}$

$\hat{e}$ is a unit vector.

Then $H' = \bar{B} \cdot (\bar{S}_1 + \bar{S}_2) = \bar{B} \cdot (\bar{S}_{tot})$

$= B\,\hat{e} \cdot \bar{S}_{tot}$.

$\therefore$

$[H', \hat{n} \cdot \bar{S}_{tot}] = B[\hat{e} \cdot \bar{S}_{tot}, \hat{n} \cdot \bar{S}_{tot}]$

For this to be zero.

they must commute → ie

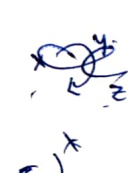
$(\hat{e} \cdot \bar{S}_{tot})(\hat{n} \cdot \bar{S}_{tot}) = (\hat{n} \cdot \bar{S}_{tot})(\hat{e} \cdot \bar{S}_{tot})$

$\Rightarrow [(e_x S^x_{tot} + e_y S^y_{tot} + e_z S^z_{tot}), (n_x S^x_{tot} + n_y S^y_{tot} + n_z S^z_{tot})] = 0$

~~$(n_x S^x_{tot} + n_y S^y_{tot} + n_z S^z_{tot})(e_x S^x_{tot} + e_y S^y_{tot} + e_z S^z_{tot})$~~

Now $[S^i, S^j] = +i\hbar\,\epsilon_{ijk} S^k$

So, It evaluates to.



$[H', \hat{n} \cdot \bar{S}_{tot}] = i\hbar [0 + e_y n_x (-1) S^z_{tot} + e_z n_x (1) S^y_{tot}$

$+ e_x n_y S^z_{tot} + 0 + e_z n_y S^x_{tot} (-1)$

$- e_x n_z S^y_{tot} + n_z e_y S^x_{tot}$

$+ 0]$

$= i\hbar [(-e_y n_x + e_x n_y) S^z_{tot}$

$+ S^y (e_z n_x - e_x n_z) S^y_{tot} + (n_z e_y - e_z n_y) S^x_{tot}]$

$\sim i\hbar [(\hat{e} \times \hat{n}) \cdot \bar{S}_{tot}]$.

So, this $= 0$

only iff

$\hat{n}$ is $\parallel^r$ to $\hat{e}$

or " " $-\hat{e}$.

So, $[H', \hat{n} \cdot \bar{S}_{tot}] \neq 0$

in general breaking rotational symmetry

So,

$$[H, \hat{n}.\bar{S}_{tot}] = [H_{rot}, \hat{n}.\bar{S}_{tot}] + [H', \hat{n}.\bar{S}_{tot}]$$

$$= 0 + 0.$$

iff when:

~~$\hat{e}$ is aligned~~

$\hat{n}$ is along / or $\hat{e}$ or $-\hat{e}$

i.e parallel / antiparallel to $\vec{B}$-axis.

Making it rotationally symmetric again.

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