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Assignment-4

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a) 8. If

$\{\bar{u}_m\}$ are basis vector functions & such that they satisfy

(1) $\int \bar{u}_m \cdot \bar{u}_n \, d^3x = \delta_{mn}$.

(2) $\bar{\nabla} \cdot \bar{u}_m(x) = 0 \quad \forall m$

(3) $\hat{n} \times \bar{u}_m = 0$.

(4) $\nabla^2 u_m = -k_m^2 u_m$

Show

$$\int (\bar{\nabla} \times \bar{u}_n) \cdot (\bar{\nabla} \times \bar{u}_m) \, d^3x = k_n^2 \delta_{nn}.$$

Ans. Here If. $\bar{E}(x,t) = \sum_m f_m(t) \bar{u}_m(\bar{x})$.

since, $\bar{\nabla} \cdot \bar{E} = 0$ (condition (2) holds).

(4) Holds as $\Rightarrow \nabla^2 \bar{E} - \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2} = 0$

(3) is imposed because the tangential component of the electric field E must vanish on a conducting surface (in vacuum)

Claim 1: $\nabla \cdot (\bar{A} \times \bar{B}) = \bar{B} \cdot (\bar{\nabla} \times \bar{A}) - \bar{A} \cdot (\bar{\nabla} \times \bar{B})$

Let us denote, $\bar{A} = \bar{u}_n$; $\bar{B} = \bar{\nabla} \times \bar{u}_m$

Then

$$\bar{B} \cdot (\bar{\nabla} \times \bar{A}) = \nabla \cdot (\bar{A} \times \bar{B}) + \bar{A} \cdot (\nabla \times \bar{B})$$

Now,

$$\bar{A} \cdot (\nabla \times \bar{B}) = \bar{u}_n \cdot (\bar{\nabla} \times (\nabla \times \bar{u}_m))$$

(By property) $= \bar{u}_n \cdot (\nabla (\nabla \cdot \bar{u}_m) - \nabla^2 u_m)$

$= \bar{u}_n \cdot (+ k_m^2 u_m) = k_m^2 \bar{u}_n \cdot \bar{u}_m$

$\therefore$

$$\int (\bar{\nabla} \times \bar{u_m}) \cdot (\nabla \times \bar{u_n})\, d^3x$$

$$= \int \nabla \cdot \left(\bar{u_n} \times (\nabla \times \bar{u_m}) \right) dV + K_m^2 \int \bar{u_n}(x) \cdot \bar{u_m}(x)\, dV$$

Claim 2: 1st integral vanishes.

$$= K_m^2 \cdot \delta_{mn}.$$

□.

Proofs.

Claim 2:

From divergence Theorem we have

$$\int \nabla \cdot (\bar{u_n} \times (\nabla \times \bar{u_m}))\, dV$$

$$= \int \bar{u_n} \times (\nabla \times \bar{u_m}) \cdot d\vec{s}$$

$$= \int \bar{u_n} \times (\nabla \times \bar{u_m}) \cdot \vec{n}\, d\vec{s}$$

observe

$$[\vec{n} \quad \bar{u_n} \quad \nabla \times \bar{u_m}]$$

$\bar{A} \cdot (\bar{B} \times \bar{C}) = [A\,B\,C]$

$$= [\bar{\nabla} \times \bar{u_m} \quad \bar{n} \quad u_n]$$ From cyclic relation.

$$= (\bar{\nabla} \times \bar{u_m}) \cdot (\bar{n} \times \bar{u_n})$$

$= 0$ by property

$= 0$

□.

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Claim 1:

$$\bar{\nabla} \cdot (\bar{A} \times \bar{B}) = \bar{B} \cdot (\nabla \times \bar{A}) - \bar{A} \cdot (\nabla \times \bar{B})$$

In index notation we can write.

$$(\bar{A} \times \bar{B})_i = \epsilon_{ijk} A_j B_k \equiv \sum_{j,k} \epsilon_{ijk} A_j B_k$$

(Einstein's convention)

$$\therefore \bar{\nabla} \cdot (\bar{A} \times \bar{B}) = \frac{\partial}{\partial x_i} (\epsilon_{ijk} A_j B_k)$$

$$= \epsilon_{ijk} \frac{\partial A_j}{\partial x_i} B_k + \epsilon_{ijk} A_j \frac{\partial B_k}{\partial x_i}$$

$i \to k$, $j \to j$

Levi-Civita property.

$$= B_k \left(\epsilon_{kij} \frac{\partial A_j}{\partial x_i} \right) - A_j \left(\epsilon_{jik} \frac{\partial B_k}{\partial x_i} \right)$$

$$= \bar{B} \cdot (\bar{\nabla} \times \bar{A}) - \bar{A} \cdot (\nabla \times \bar{B})$$

□