Assignment-1

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Q1) Given

$$f(x) = \begin{cases} +1 & x \geq 0 \\ -1 & x < 0 \end{cases}$$

(a) Express $f(x)$ as a sum of Legendre Polynomials $P_\ell(x)$ in interval $[-1,1]$

(b) Establish a relation b/w x^ℓ and $P_\ell(x)$ for $\ell = 1, 3, 5$

(c) Write step f^n in as $\sum a_n x^n$ accurate upto 5^{th} order.

Also plot this with pen and paper.

(a) Let $f(x) = \sum_{n=0}^{\infty} a_n P_n(x)$ (Since, Legendre Poly are complete in $[-1,1]$)

Multiply by $P_m(x)$ both sides and integrate in $[-1,1]$

$$\int_{-1}^{1} f(x) P_m(x)\, dx = \sum_{n=0}^{\infty} \int_{-1}^{1} dx\, a_n P_n(x) P_m(x)$$

From orthogonality relations we have,

$$\text{RHS} = \sum_{n=0}^{\infty} a_n \left(\frac{2}{2n+1} \delta_{mn} \right)$$

$$= \frac{2}{2m+1} a_m$$

$$\therefore a_m \frac{2}{2m+1} = \int_0^1 P_m(x)\, dx - \int_{-1}^{0} P_m(x)\, dx$$

Subst. $x = -y$

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$$= \int_0^1 dx\ P_m(x) + \int_1^0 P_m(-y)\,dy$$

$$\because P_m(-y) = (-1)^m P_m(y) \quad \forall m,$$

$$= \int_0^1 dx\ P_m(x) + \int_0^1 (-1)^{m+1} P_m(y)\,dy.$$

$$= \int_0^1 [1 + (-1)^{m+1}]\, P_m(y)\,dy$$

For m = even

$$1 + 1 = 0 \qquad \therefore \boxed{a_m = 0}$$

For m = odd.

$$\frac{2}{2m+1} a_m = 2 \int_0^1 P_m(x)\,dx.$$

$$\left[\frac{a_m}{2m+1} = \int_0^1 P_m(x)\,dx\right] \qquad m = 1, 3, 5, 7, \ldots$$

From Rodrigues' Formula.

$$P_m(x) = \frac{1}{2^m\, m!} \frac{d^m}{dx^m} (x^2-1)^m$$

$\frac{2m-m+1}{2}$ $\quad \frac{m-1}{2}$

$$\Rightarrow (x^2-1)^m = {}^mC_0 (x^2)^m + {}^mC_1 (x^2)^{m-1} (-1)^1 + {}^mC_{\left(\frac{m+1}{2}\right)} (x^2)^{\frac{m+1}{2}} (-1)^{\left(\frac{m-1}{2}\right)} + {}^mC_{\left(\frac{m-1}{2}\right)} (x^2)^{\frac{m-1}{2}} (-1)^{\left(\frac{m+1}{2}\right)} + {}^mC_{\left(\frac{m+3}{2}\right)} (x^2)^{\frac{m+3}{2}} (-1)^{\frac{m+3}{2}}$$

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∴ RHS we have,

$$\frac{1}{2^m m!} \int_0^1 1 \times D^m (x^2-1)^m \, dx$$

By parts

$$= \frac{1}{2^m m!} \left(D^{m-1} (x^2-1)^m \Big|_0^1 - 0 \right)$$

Now:

$$D^{m-1}(x^2-1)^m = (x^2-1) f(x)$$

for some $f(x) \in \mathbb{P}_{2n}[x]$

$$\therefore \left((x^2-1) f(x) \right) \Big|_1 = 0$$

$$\therefore = \frac{-1}{2^m m!} \left(\frac{d^{m-1}}{dx^{m-1}} (x^2-1)^m \right) \Big|_{x=0}$$

Now expanding

$$\frac{d^{m-1}}{dx^{m-1}} (x^2-1)^m = \binom{m}{0} 2m(2m-1)\cdots(m+2) X^{m+1}$$

$$- \binom{m}{1} (2m-2)(2m-3)\cdots m \, X^{m-1} + \cdots$$

$$+ \binom{m}{\frac{m+1}{2}} (m+1)(m)\cdots 3 \, x^2 (-1)^{\frac{m-1}{2}}$$

$$+ \binom{m}{\frac{m-1}{2}} (m-1)! \, (-1)^{\frac{m+1}{2}} + 0$$

since $\frac{d^{m-1}}{dx^{m-1}} (x^j) = 0 \quad \forall j < m-1$

evaluating at $x = 0$ give,

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RHS

$$= \frac{-1}{2^m \, m!} \binom{m}{\frac{m-1}{2}} (m-1)! \, (-1)^{\frac{m+1}{2}}$$

$\therefore$

$$a_m = (2m+1) \frac{-1}{2^m \cdot m!} \frac{m!}{\left(\frac{m+1}{2}\right)! \left(\frac{m-1}{2}\right)!} (m-1)! \, (-1)^{\frac{m+1}{2}} \quad \ldots\ldots (*)$$

$$= (2m+1) \frac{-1}{2^m} \frac{1}{\left(\frac{m+1}{2}\right)! \left(\frac{m-1}{2}\right)!}$$

m = odd

$$\left\lfloor \frac{m+1}{2} \right\rfloor = \frac{\frac{m-1}{2} \cdot \frac{m-3}{2} \cdots \frac{4}{2} \cdot \frac{2}{2}}{\left(\frac{m-1}{2}\right)!}$$

$$= \frac{(m-1)(m-3)\cdots 4 \cdot 2}{2^{\frac{m-1}{2}}}$$

Furth $(-1)^{\frac{m+1}{2}} = -(-1)^{\frac{m-1}{2}}$

$\therefore$

$$a_m = (2m+1) \frac{-1}{2^m} \left(\frac{1}{\left\lfloor \frac{m+1}{2} \right\rfloor} \right) \frac{2^{\frac{-m-1}{2}}}{(m-1)(m-3)\cdots 4 \cdot 2} (m-1)! \, (-1) \, (-1)^{\frac{m-1}{2}}$$

$$2^{\frac{-m-1}{2}} = \frac{2^{\frac{-m+1}{2}}}{2}$$

$$\therefore a_m = \frac{(2m+1)}{\left\lfloor \frac{m+1}{2} \right\rfloor} \left(\frac{-1}{2}\right)^{\frac{m-1}{2}} \frac{1}{2} \cdot \frac{(m-1)(m-2)\cdots 4 \cdot 3 \cdot 2 \cdot 1}{(m-1)(m-3)\cdots 4 \cdot 2}$$

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Thus

$$a_m = \left(\frac{-1}{2}\right)^{\frac{m-1}{2}} \frac{1}{2} \cdot \frac{(2m+1)}{\left(\frac{m+1}{2}\right)!} (m-2)!!$$

So,

$$H(x) = \sum_{m=1,3,5,\ldots}^{\infty} a_m P_m(x) \quad \text{in } [-1,1]$$

(b) For $m = 1$

$$a_1 = 1 \cdot \frac{1}{2} \cdot \frac{3}{1} \cdot 1 = \frac{3}{2}$$

For $m = 3$

$$a_3 = \left(\frac{-1}{2}\right) \frac{1}{2} \cdot \frac{7}{2} \cdot 1 = -7/8$$

For $m = 5$

$$a_5 = \left(\frac{-1}{2}\right)^2 \frac{1}{2} \cdot \frac{11}{62} \cdot 3 \times 1 = \frac{11}{16}$$

$\therefore$

$$H(x) = \frac{3}{2} P_1(x) + \frac{-7}{8} P_3(x) + \frac{11}{16} P_5(x) + \ldots$$

From Rodrigues' Formula, we can derive,

$$P_1(x) = x \quad ; \quad P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$

(3) 1, 3, 5

2 × 3 + 1

(7)

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Putting,

Let us denote

$$\textcircled{H}_K := \sum_{l=1,3,\ldots K} a_l P_l(x)$$

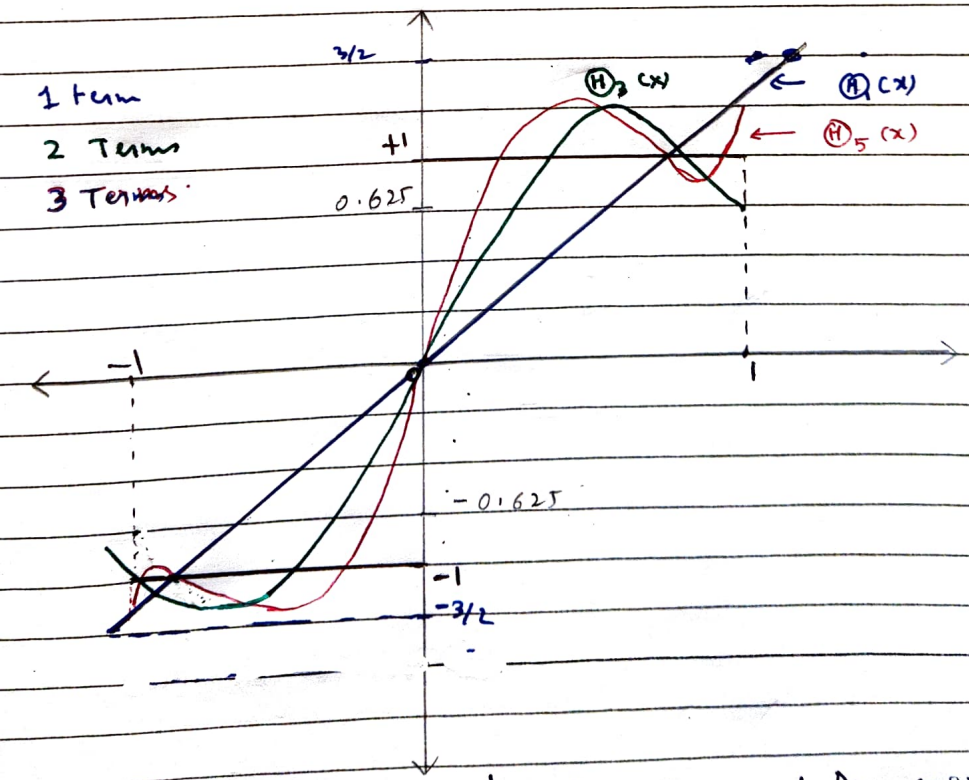
$$\textcircled{H}_1(x) = \frac{3}{2} x .$$

$$\textcircled{H}_3(x) = \frac{3}{2} P_1(x) - \frac{7}{3} P_3(x)$$

$$= \frac{5x}{16} (9 - 7x^2) .$$

$$\textcircled{H}_5(x) = \frac{3}{2} P_1(x) - \frac{7}{8} P_3(x) + \frac{11}{16} P_5(x)$$

$$= \frac{21x}{128} (33x^4 - 50x^2 + 25) .$$



Plot of Legendre Polynomial Decomposition of step function $\textcircled{H}(x)$