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PH3103 HW-5

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① $f(z) = \dfrac{1}{(z-2)^2}$

A Since $z=0$ is a point in the domain of analyticity of $f(z)$, we can expand it in the form of taylor series about $z=0$.

$$\therefore \quad f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

$$a_n = \frac{1}{n!} \frac{d^n f}{dz^n}\bigg|_{z=z_0}$$

$$a_0 = f(0) = \frac{1}{4}$$

$$a_1 = f'(0) = \frac{-2}{(z-2)^3}\bigg|_{z=0} = \frac{+2}{-8} = \frac{1}{4}$$

$$a_2 = \frac{1}{2!} f''(0) = \frac{6}{(z-2)^4} \times \frac{1}{2}\bigg|_{z=0} = \frac{3}{16} \cdots$$

Notice.

$$f^{(n)}(z) = \frac{(-1)^n \, 1.2.3 \cdots (n+1)}{(z-2)^{n+2}}$$

$$\therefore \quad a_n = \frac{1}{n!} f^{(n)}(0) = \frac{1}{n!} (-1)^n \frac{(n+1)!}{(-2)^{n+2}}$$

$\therefore$ For $f(z) = \sum a_n z^n$

By cauchy theorem on limits, radius of convergence R, is given by.

(i) $\dfrac{1}{R} = \lim |a_n|^{1/n}$ (ii) $\lim \left|\dfrac{a_{n+1}}{a_n}\right| = \dfrac{1}{R}$.

In our case,

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{1}{(n+1)!} \frac{f^{(n+1)}(0)}{f^{(n)}(0)} \, n!\right| = \left|\frac{1}{n+1} \frac{(-1)^{n+1}(n+2)!}{(-2)^{n+3}} \frac{(-2)^{n+2}}{(-1)^n (n+1)!}\right|$$

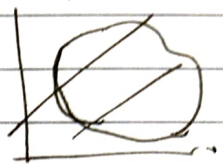
Simplifying.

$$= \left| \frac{n+2}{n+1} \cdot \frac{1}{2} \right| = \left| \frac{1+\frac{2}{n}}{1+\frac{1}{n}} \right| \frac{1}{2} \to \frac{1}{2}$$

$\therefore \boxed{R = 2}$

Ans.

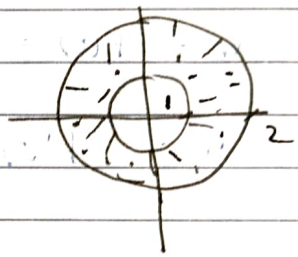
(2) $$f(z) = \frac{1}{(z-2)(z-1)}$$



(i) $1 < |z| < 2$

~~In this region f(z) is analytic.~~

$f(z)$ has poles at $z=1$ and $z=2$.



$$f(z) = \frac{(z-1)-(z-2)}{(z-2)(z-1)}$$

$$= \frac{1}{z-2} - \frac{1}{z-1} \qquad \left|\frac{z}{1}\right| < 1 \text{ or } \left|\frac{1}{z}\right| < 1$$

↓

can be expanded as binomial series.

either

$$\left|\frac{z}{2}\right| < 1 \text{ or } \left|\frac{2}{z}\right| < 1$$

In this region, $1 < |z|$ & $|z| < 2$.

$\therefore$

$$f(z) = \frac{1}{z-2} - \frac{1}{z-1}$$

$$= \frac{1}{-2\left(1-\frac{z}{2}\right)} - \frac{1}{z\left(1-\frac{1}{z}\right)}$$

$$= -\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$

Now, we can carry out expansion as it satisfies the region.

$$f(z) = \frac{-1}{2}\left(1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \cdots\right) - \frac{1}{z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \cdots\right)$$

$$f(z) = \frac{-1}{2}\left(1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \cdots\right) + \left(\frac{-1}{z} - \frac{1}{z^2} - \frac{1}{z^3} - \cdots\right)$$

we are expanding about $z = 0$

$\therefore \text{Res}(f(z))\big|_{z=0} = a_{-1} = -1$

(ii) $|z| > 2 \rightarrow \left|\frac{2}{z}\right| < 1$

Now, similarly we take $|z|$ common.

$$f(z) = \frac{1}{z}\left(1 - \frac{2}{z}\right)^{-1} - \frac{1}{z}\left(1 - \frac{1}{z}\right)^{-1}$$

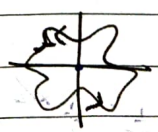
$$= \frac{1}{z}\left(1 + \frac{2}{z} + \frac{4}{z^2} + \frac{8}{z^3} + \cdots\right) - \frac{1}{z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \cdots\right)$$

$$= \frac{1}{z^2} + \frac{3}{z^3} + \frac{7}{z^4} + \cdots$$

This is the required Laurent series expansion.

$\text{Res}(f(z))\big|_{z=0} = a_{-1} = 0$!

(6) $J_1 = \oint_C \frac{e^z}{z^3}\, dz$



$f(z) = \frac{e^z}{z^3}$ is analytic at all points except at $z = 0$.

So, by Residue theorem we have:

$$\oint_C f(z)\,dz = 2\pi i \sum_i \text{Res}\,(f(z))\Big|_{z=z_i}$$

$$= 2\pi i \;\text{Res}\, f(z)\Big|_{z=0}$$

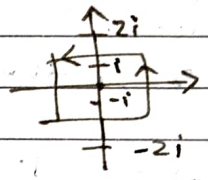
$$\frac{e^z}{z^3} = \frac{1}{z^3}\left(1 + z + \frac{z^2}{\lfloor 2} + \frac{z^3}{\lfloor 3} + \cdots\right)$$

$$= \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z \times 2} + \frac{1}{\lfloor 3} + \cdots$$

$$\therefore a_{-1} = \frac{1}{2} \qquad I_1 = 2\pi i \times \frac{1}{2} = \boxed{\pi i}$$

$\therefore$

(7) $$I_2 = \oint_C \frac{\cos z}{z(z^2+8)}\,dz$$



Inside C, bad point at $z = 0$

$\therefore \oint_C f(z)\,dz = 2\pi i \;\text{Res}\, f(z)\Big|_{z=0}$.

since $|z| < 2\sqrt{2}$ we can safely do binomial exp.

$$\frac{\cos z}{z} = \left[\frac{1}{z} - \frac{z}{\lfloor 2} + \frac{z^3}{9!} \cdots\right]$$

$$\frac{1}{z^2+8} = \frac{1}{2} \cdot \frac{1}{2\sqrt{2}i} \frac{(z+2\sqrt{2}i) - (z-2\sqrt{2}i)}{(z-2\sqrt{2}i)(z+2\sqrt{2}i)}$$

$$= \frac{1}{2} \cdot \frac{1}{2\sqrt{2}i}\left[\frac{-1}{2\sqrt{2}i}\left(1 - \frac{z}{2\sqrt{2}i}\right)^{-1} - \frac{1}{2\sqrt{2}i}\left(1 + \frac{z}{2\sqrt{2}i}\right)^{-1}\right]$$

$$= \frac{1}{2} \cdot \frac{1}{2\sqrt{2}i} \times \frac{1}{2\sqrt{2}i}\left[1 + \frac{z}{2\sqrt{2}i} + \left(\frac{z}{2\sqrt{2}i}\right)^2 + \cdots\right.$$
$$\left. 1 - \left(\frac{z}{2\sqrt{2}i}\right) + \left(\frac{z}{2\sqrt{2}i}\right)^2 - \cdots\right\}$$

$\therefore$

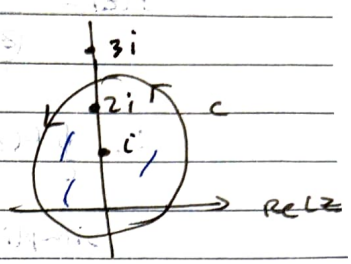
$$f(z) = \frac{1}{2} \cdot \frac{1}{8} \times 2\left\{1 + \left(\frac{z}{2\sqrt{2}i}\right)^2 + \cdots\right\}\left\{\frac{1}{z} - \frac{z}{\lfloor 2} + \frac{z^3}{4!} \cdots\right\}$$

$$\boxed{a_{-1} = \frac{1}{8}}$$

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$I_2 = 2\pi i \times \frac{1}{8i} = i\frac{\pi}{4}$ Ans

(8) $I_3 = \oint_C \frac{dz}{(z^2+4)^2}$



$f(z)$ is analytic at all points, except at $z = 2i$

$\therefore \oint_C f(z) = 2\pi i \cdot \text{Res } f(z)\big|_{z=2i}$

~~$= 2\pi i \times \frac{1}{16}$~~

$\frac{1}{(z^2+4)^2} = \frac{1}{(z-2i)^2}\left[\frac{1}{(z-2i)+4i}\right]^2$

subst. let $w = z - 2i$

Notice. we are in region $|z-i| < 2$

$\therefore |w| = |z-i-i| \le |z-i| + |-i|$

$< 2 + 1 = 3 < 4.$

$= \frac{1}{w^2}\left[\frac{1}{w+4i}\right]^2$

∴ since $|w| < 4$ we can do Binomial exp

$= \frac{1}{w^2}\cdot\frac{1}{16}\cdot\left(1+\frac{w}{4i}\right)^{-2}$

$= \frac{-1}{16w^2}\left(1 - \frac{2w}{4i} + \frac{2\cdot 3}{2}\left(\frac{w}{4i}\right)^3\right)$

$\therefore a_{-1} = \frac{1}{16}\cdot\frac{1}{2i}$

$\therefore I_3 = 2\pi i \times \frac{1}{2i}\cdot\frac{1}{16} = \boxed{\frac{\pi}{16}}$ Ans

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⑨ $I_4 = \oint_C \frac{z}{z^2+9} \cdot dz$

$f(z) = \frac{z}{(z-2i)(z+2i)}$ Pole at $z = 2i, -2i$

Both 1st order.

$\therefore \oint_C f(z)\,dz = 2\pi i \times \left[\text{Res } f(z)\Big|_{z=2i} + \text{Res } f(z)\Big|_{z=-2i} \right]$

Simplifying:

$f(z) = \frac{1}{2(z+2i)} + \frac{1}{2(z-2i)}$

Region $|z| < 3$

Around $z = 2i$

~~subst $w = z - 2i$~~

we use,

$a_{-1} = \text{Res } f(z)\Big|_{z=z_0} = \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} \left[(z-z_0)^n f(z) \right]\Big|_{z=z_0}$

$n=1$ for both

$= \lim_{z_0 \to 2i} \frac{(z-2i)\, z}{(z-2i)(z+2i)}$

$= \frac{2i}{4i} = \frac{1}{2}$.

similarly at $z = -2i$

$a_{-1} = \lim_{z \to -2i} \frac{(z+2i)\, z}{(z-2i)(z+2i)} = \frac{-2i}{-4i} = \frac{1}{2}$.

$\therefore I_4 = 2\pi i \left(\frac{1}{2} + \frac{1}{2} \right) = 2\pi i$ □

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③ (b) $f(z) = \dfrac{1}{z \sin(\pi z)(z+1/2)}$.

We find the poles, by equating D^r to zero

Claim
- $z_0 = 0 \rightarrow$ 2nd order pole.
- $z_0 = k$, $k = \pm 1, \pm 2, \ldots \rightarrow$ 1st order pole.
- $z_0 = -1/2 \rightarrow$ 1st order pole.

For $\boxed{z_0 = -\frac{1}{2}}$

$$\text{Res } f(z)\Big|_{z_0} = \lim_{z \to -\frac{1}{2}} (z+\tfrac{1}{2}) \cdot \frac{1}{z \sin(\pi z)(z+\tfrac{1}{2})}$$

$$= \frac{1}{-1^2} = \boxed{2}.$$

For, $\boxed{z_0 = k}$, $k = \pm 1, \pm 2, \pm 3, \ldots$

Consider

$f(z) = \dfrac{g(z)}{h(z)}$ $\quad g(z) = \dfrac{1}{z(z+1/2)}$ $\quad h(z) = \sin \pi z$.

$g(k) \neq 0$ analytic!

$\dfrac{dh}{dz} = \pi \cos \pi z \neq 0$ at $z = k$.

$\therefore$ $h(z)$ has a simple zero

So, $\text{Res } f(z)\Big|_{z_0} = \dfrac{g(z_0)}{h'(z_0)}$

$$= \lim_{z \to k} \frac{\frac{1}{z(z+1/2)}}{\pi \cos(\pi z)}$$

$$= \boxed{\frac{(-1)^k}{k(k+1/2)} \cdot \frac{1}{\pi}}$$

Ans.

To investigate $\boxed{z_0 = 0}$, we expand about $z=0$.

$$f(z) = \frac{1}{\sin \pi z}\left[\frac{1}{z\cdot(z+\frac{1}{2})}\right]$$

$$= \frac{4}{\sin \pi z}\left[\frac{1}{2z} - \frac{1}{2}(1+2z)^{-1}\right]$$

Domain of validity of expansion: $|z| < 1/2$

$$= 4\left[\frac{1}{2z} - \frac{1}{2}\left(1-2z+4z^2+\cdots\right)\right]\left[\pi z - \frac{(\pi z)^3}{3!} + \cdots\right]^{-1}$$

$$= \frac{4}{\pi z}\left[\frac{1}{2z} - \frac{1}{2}\left(1-2z+4z^2+\cdots\right)\right]\left[1 + \frac{(\pi z)^2}{3!} - \cdots\right]$$

↳ Max^m inverse power $\sim \frac{1}{z^2}$

$\therefore$ $\boxed{\text{order of pole} = 2}$

$$\therefore \operatorname{Res} f(z)\Big|_{z=0} = a_{-1} = \frac{4}{\pi}\cdot\frac{1}{2}\cdot\left(-\frac{1}{2}\right)\cdot 1$$

$$\boxed{a_{-1} = \frac{-4}{\pi}}$$ Ans.

(ii) $f(z) = \operatorname{cosec}(\pi z)$.

$$f(z) = \frac{1}{\sin(\pi z)}$$

Poles: $z = 0$

$z = k$, $k = \pm n$ $(n \in \mathbb{N})$

Notice, $f(z)$ has 1st order pole at k, $k = 0, \pm 1, \pm 2, \ldots$

Consider $w := z - k$

$$f(w) = \frac{1}{\sin(\pi w + k\pi)}$$

let $\alpha = \pi w + k\pi$

$$= \frac{1}{\alpha - \frac{\alpha^3}{\lfloor 3} + \frac{\alpha^5}{\lfloor 5} - \cdots} = \frac{1}{\alpha\left(1 - \frac{\alpha^2}{\lfloor 3} + \frac{\alpha^4}{\lfloor 5} - \cdots\right)}$$

$$= \frac{1}{\alpha}\left[1 - \frac{\alpha^2}{\lfloor 3} + \frac{\alpha^4}{\lfloor 5} - \cdots\right]^{-1}$$

$$= \frac{1}{\alpha}\left[1 + \frac{\alpha^2}{\lfloor 3} - \frac{\alpha^4}{\lfloor 5} \cdots\right]$$

Highest negative power $= \frac{1}{\alpha} = \frac{1}{\pi w + k\pi}$.

$\therefore$ order of pole at $= 1$ for all cases.

$\therefore$

$$\operatorname{Res} f(z)\Big|_{z=k} = \lim_{z \to k} \frac{z-k}{\sin \pi z}$$ L'H rule,

$$= \lim_{z \to k} \frac{1}{\pi \cos \pi z} = \boxed{\frac{(-1)^k}{\pi}}$$

$k \in \mathbb{Z}$

$k \in \mathbb{Z}$

(9) Given $f(z)$ is entire that is, analytic in entire $\mathbb{C}$.

⇒ Let $z_0 \in \mathbb{C}$, be any number.

$f(z)$ is analytic in the entire region within $|z - z_0| = R$ with $R \to \infty$

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From Cauchy's Formula,

$$f^{(1)}(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)\,dz}{(z-z_0)^2}$$

$$|f^{(1)}(z_0)| = \left| \frac{1}{2\pi i} \oint_C \frac{f(z)\,dz}{(z-z_0)^2} \right|$$

$$\leq \frac{1}{2\pi} \oint_C \left| \frac{f(z)}{(z-z_0)^2} \right| |dz|$$

(Darboux Inequality).

Given $f(z)$ is bounded, i.e $\exists\ M \in \mathbb{R}^+$ s.t $|f(z)| \leq M$

$$\therefore \leq \frac{1}{2\pi} \frac{M}{R^2} \oint_C |dz|$$

Let $z - z_0 = Re^{i\theta}$

$$|dz| = |Ri\,e^{i\theta}\,d\theta| = R\,d\theta$$

$$= \frac{1}{2\pi} \frac{M}{R^2} \int_0^{2\pi} R\,d\theta$$

$$|f'(z_0)| \leq \frac{M}{R}$$

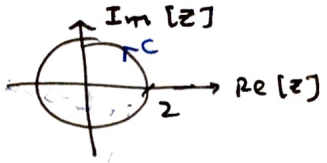
Take $R \to \infty$ (valid for any +ve R)

$$\Rightarrow |f(z)| \leq 0$$

$$\Rightarrow \boxed{f'(z_0) = 0}$$

Since $z_0 \in \mathbb{C}$ is arbitrary, we conclude a bounded entire function is constant.

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$C: |z| = 2$

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(5) (a) $\oint_C \frac{\cos z}{z^3}\, dz$.

$\frac{\cos z}{z^3}$ has only 1 bad pt. at $z = 0$ inside C.

$\therefore \oint_C f(z)\, dz = 2\pi i \times \text{Res}\, f(z)\Big|_{z=0}$.

$$\frac{\cos z}{z^3} = \frac{1}{z^3}\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots\right) = \frac{1}{z^3} - \frac{1}{2z} + \frac{z}{4!} \cdots$$

$$= \sum_{k=-\infty}^{\infty} a_k (z)^k$$

$\therefore a_{-1} = -\frac{1}{2}$ ← Residue.

$\therefore I = 2\pi i \left(-\frac{1}{2}\right) = \boxed{-i\pi}$ Ans

(b) $\oint_C \frac{dz}{z+1}$.

$\frac{1}{z+1}$ has bad pt. at $z_0 = -1$

Substitute $w := z+1$

Will do Laurent expansion about $w = 0$.

$\therefore \frac{1}{z+1} = \frac{1}{w}$. $\therefore a_{-1} = 1$

Hence, $I = 2\pi i \times 1 = \boxed{2\pi i}$ Ans

(c) $\oint_C \frac{z^2 - 2}{z(z-1)(z+4)}\, dz = \oint_C f(z)\, dz$

Notice,

$f(z)$ has bad pts at, $z = 0$, $z = 1$ inside C.

$\therefore I = 2\pi i \left[\text{Res}\, f(z)\Big|_{z=0} + \text{Res}\, f(z)\Big|_{z=1}\right]$

We know here, $f(z) = \frac{\phi(z)}{(z - z_0)^n}$ $(z \neq z_0)$

where $\phi(z)$ is analytic around $z = z_0$

in L-series exp. of $f(z)$ around $z = z_0$.

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we know.

$$\text{Res } f(z)\Big|_{z=z_0} = a_{-1} = \frac{1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}\left[(z-z_0)^n f(z)\right]$$

where 'n' is the order of the pole at $z = z_0$.

In this question,

$$\frac{z^2-2}{z(z-1)(z+4)}$$

Poles.

$z = 0 \rightarrow$ order = 1

$z = 1 \rightarrow$ order = 1

$$\therefore \text{Res } f(z)\Big|_{z=0} = \lim_{z\to 0} z \cdot \frac{z^2-2}{z(z-1)(z+4)} = \frac{1}{2}$$

$$\text{Res } f(z)\Big|_{z=1} = \lim_{z\to 1} (z-1)\cdot\frac{z^2-2}{z(z-1)(z+4)} = \frac{-1}{5}$$

So, $I = 2\pi i\left(\frac{1}{2} - \frac{1}{5}\right) = \boxed{i\frac{3\pi}{5}}$ Ans

(d) $\oint_C \frac{z}{(z-1)^3(z+3)}\,dz$

Here we have ~~sing~~ pole @ $z = 1$ of order = 3.

$$\therefore I = 2\pi i \,\text{Res } f(z)\Big|_{z=1}$$

$$a_{-1} = \frac{1}{2}\frac{d^2}{dz^2}\left[(z-1)^3\cdot\frac{z}{(z-1)^3(z+3)}\right]\Bigg|_{z=1}$$

$$= \frac{1}{2}\lim_{z\to 1}\left[\frac{-6}{(z+3)^4}\right] = \frac{-3}{4^3}$$

$$\therefore I = 2\pi i \times \frac{-3}{2^6 5}$$

$$\boxed{I = \frac{-3}{32}\pi i}$$ Ans

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10) (a) $I = \int_{-\infty}^{\infty} \frac{\sin x}{1+x^2}\,dx$.

M-1

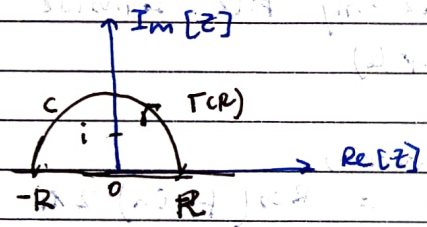
The function is odd, ∴ (By Queen Rule)

$$\therefore \int_{-\infty}^{\infty} \frac{\sin x}{1+x^2}\,dx = 0 \;//$$

M-2

We note,

$$\int_{-\infty}^{\infty} dx\, \frac{\sin x}{1+x^2} = \text{Im}\left[\int_{-\infty}^{\infty} \frac{e^{iz}}{1+x^2}\right]$$



We ~~evaluate~~ evaluate, along a ~~curve~~ contour C.

S.t

$$\oint_C \frac{e^{iz}}{z^2+1}\,dz = \lim_{R\to\infty} \int_{-R}^{R} \frac{e^{iz}}{z^2+1}\,dz + \left(\int_{\Gamma(R)} \frac{e^{iz}}{z^2+1}\,dz\right)$$

↓ This exactly the X-axis

↓ = 0 By Jordan's Lemma.

$$\therefore \int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+1}\,dx = \oint \frac{e^{iz}}{z^2+1}\,dz$$

$$= 2\pi i \times \text{Res } f(z)\Big|_{z=i}$$ ← 1st order pole at z=0

$$= 2\pi i \left[\lim_{z\to i} \frac{(z-i)\,e^{iz}}{(z-i)(z+i)}\right]$$

$$= 2\pi i \cdot \frac{e^{-1}}{2i} = \frac{\pi}{e}$$

∴ Req. $I = \text{Im}\left(\frac{\pi}{e}\right) = 0$! Hence verified.

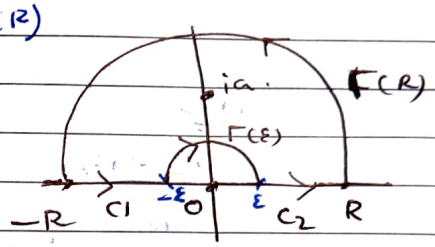
10) (b) $\int_0^\infty \frac{\sin x}{x(x^2+a^2)}\,dx \qquad a>0.$

$$I = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\sin x}{x(x^2+a^2)}\,dx = \frac{1}{2}\,\mathrm{Im}\left\{\int_{-\infty}^{\infty} \frac{e^{ix}}{x(x^2+a^2)}\,dx\right\}$$

Consider the following $\oint$-contour

$$\Gamma = C_1 + \Gamma(\varepsilon) + C_2 + \Gamma(R)$$

$$f(z) = \frac{e^{iz}}{z(z^2+a^2)}$$



We shall take the limit

$R \to \infty, \quad \varepsilon \to 0^+$

So, $z = ia$ is the only pole inside contour (order = 1 ; simple pole)

∴ By Residue Theorem,

$$\lim_{\substack{\varepsilon\to 0^+ \\ R\to\infty}} \oint_\Gamma f(z)\,dz = \mathrm{Res}(f; ia)\, 2\pi i$$

Now,

$$\mathrm{Res}(f; ia) = \lim_{z\to ia}\left[(z-ia) f(z)\right]$$

$$= \lim_{z\to ia}\left[\frac{(z-ia)\, e^{iz}}{(z-ia)\, z\,(z+ia)}\right]$$

$$= \frac{e^{-a}}{ia(2ia)} = \frac{-e^{-a}}{2a^2}$$

$$\therefore \oint_\Gamma f(z)\,dz = -2\pi i \cdot \frac{e^{-a}}{2a^2} = \frac{-\pi i}{a^2} e^{-a}$$

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$$\oint_{\Gamma} f = \int_{C_1} + \int_{C_2} + \int_{\Gamma(\varepsilon)} + \int_{\Gamma(R)} \quad \cdots (*)_1$$

For $\Gamma(R)$ contour we can apply Jordan's Lemma

$$\text{i.e} \quad \lim_{R\to\infty} \int_{\Gamma(R)} f(z)\,dz = 0 .$$

M-1

Now see

$$\lim \int_{C_1}^{\infty} + \int_{C_2} = \lim_{\substack{\varepsilon\to 0^+ \\ R\to\infty}} \left[\int_{-R}^{-\varepsilon} f(x)\,dx + \int_{\varepsilon}^{R} f(x)\,dx \right]$$

$\because$ on x-axis contour!

$$= P \int_{-\infty}^{\infty} \frac{e^{ix}}{x(a^2+x^2)}\,dx := I'$$

principal value

Consider $\Gamma(\varepsilon)$

$z = \varepsilon e^{i\theta} \quad 0 \le \theta \le \pi$ (clockwise)

$$\therefore \lim_{\varepsilon\to 0^+} \int_{\Gamma(\varepsilon)} f(z)\,dz = \lim_{\varepsilon\to 0^+} \int_{\pi}^{0} \frac{\exp(i\varepsilon e^{i\theta})}{\varepsilon e^{i\theta}\cdot(\varepsilon^2 e^{2i\theta}+a^2)}\, i\varepsilon e^{i\theta}\,d\theta$$

$$= \int_{\pi}^{0} \lim_{\varepsilon\to 0^+} i\,\frac{\exp(i\varepsilon e^{i\theta})}{(\varepsilon^2 e^{2i\theta}+a^2)}\,d\theta$$

$$= -\frac{\pi i}{a^2}$$

From $(*)_1$

$$-\frac{\pi i}{a^2} e^{-a} = I' + -\frac{\pi i}{a^2} \Rightarrow I' = \frac{\pi i}{a^2}(1-e^{-a})$$

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So Final Ans

$$\int_0^\infty \frac{\sin x}{x(x^2+a^2)}\,dx = \frac{1}{2}\,\mathrm{Im}(I_\theta') = \boxed{\frac{\pi}{2a^2}(1-e^{-a})} \qquad a>0$$

Ans

M-2

Observe

$$\lim_{\substack{\varepsilon\to 0+\\ R\to\infty}} \int_{C_1} + \int_{C_2} = \lim_{\substack{\varepsilon\to 0+\\ R\to\infty}} \int_{-R}^{-\varepsilon} \frac{e^{ix}}{x(x^2+a^2)}\,dx + \int_{\varepsilon}^{R} \frac{e^{ix}}{x(x^2+a^2)}\,dx$$

Subst. $x \to -x$

$$= \int_{R}^{\varepsilon} \frac{e^{-ix}\,dx}{x(x^2+a^2)}$$

$$\therefore = \lim_{\substack{\varepsilon\to 0+\\ R\to\infty}} \int_{\varepsilon}^{R} \frac{e^{ix}-e^{-ix}}{x(x^2+a^2)}\,dx = 2i\int_0^\infty \frac{\sin x}{x(x^2+a^2)}\,dx$$

$$= 2iI$$

$$\therefore \quad \frac{-\pi i\, e^{-a}}{a^2} = 2iI - \frac{\pi i}{a^2} \Rightarrow \boxed{I = \frac{\pi}{2a^2}(1-e^{-a})}$$

Ans

10) © $\int_0^{2\pi} \frac{\sin^2\theta}{a+b\cos\theta}\, d\theta \qquad a > b > 0$.

consider the integrand,

$$f(\theta) = \frac{1}{b^2}\frac{b^2\sin^2\theta}{a+b\cos\theta} = \frac{1}{b^2}\frac{b^2-b^2\cos^2\theta}{a+b\cos\theta} = \frac{1}{b^2}\frac{a^2-b^2\cos^2\theta+b^2-a^2}{a+b\cos\theta}.$$

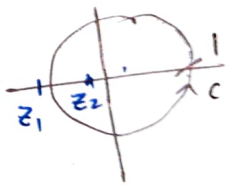
Since $a > b > 0 \therefore a^2-b^2 > 0$ and $a+b\cos\theta \neq 0$ $\forall \theta \in [0, 2\pi]$

$$\therefore f(\theta) = \frac{1}{b^2}(a-b\cos\theta) + \frac{b^2-a^2}{b^2}\frac{1}{a+b\cos\theta}$$

$$I = \int_0^{2\pi} f(\theta)\, d\theta = \int_0^{2\pi} \left\{ \left[\frac{1}{b^2}(a-b\cos\theta)\right] + \left(\frac{b^2-a^2}{b^2}\right)\frac{1}{a+b\cos\theta} \right\} d\theta$$

$$= \int_0^{2\pi} \frac{a}{b^2}\, d\theta - \frac{1}{b}\int_0^{2\pi} \cos\theta\, d\theta + \frac{b^2-a^2}{b^2}\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta}$$

$$\therefore I = \frac{a}{b^2}2\pi - \left(\frac{a^2-b^2}{b^2}\right)\underbrace{\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta}}_{\equiv I'}$$



Now,

$$I' = \int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \int_0^{2\pi} \frac{d\theta}{a+b\left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)}$$

Consider a contour C of (unit circle),

Let, $z(\theta) = e^{i\theta} \quad ; 0 \le \theta < 2\pi$

$$\frac{dz}{i} = e^{i\theta}.d\theta = z\, d\theta$$

$$\therefore I' = \oint_C \frac{dz}{iz}\frac{1}{a+\frac{b}{2}\left(z+\frac{1}{z}\right)} = -2i\oint_C \frac{dz}{bz^2+2az+b} = -\frac{2i}{b}\oint_C \frac{dz}{z^2+2\alpha z+1}$$

$$\boxed{\alpha = \frac{a}{b} > 1}$$

$$= \frac{-2i}{b} \oint_C \frac{dz}{(z+\alpha)^2 - (\alpha^2-1)} = \frac{-2i}{b} \oint_C \frac{dz}{\left(z+\alpha+\sqrt{\alpha^2-1}\right)\left(z+\alpha-\sqrt{\alpha^2-1}\right)}$$

since, $\alpha > 1$, $\sqrt{\alpha^2-1} > 0$ ⟹ $\alpha + \sqrt{\alpha^2-1} > 1$, $\alpha - \sqrt{\alpha^2-1} < 1$

$\therefore z_1 = -\left(\alpha + \sqrt{\alpha^2-1}\right) < -1$ → simple pole not in contour

while, $z_2 = -\left(\alpha - \sqrt{\alpha^2-1}\right) > -1$ → " " in contour.

($\alpha > 0$)

Consider,

$$g(z) = \frac{1}{z^2 + 2\alpha z + 1}$$

$$\therefore \oint_C g(z) = 2\pi i \times \text{Res}(g; z_2) \quad \cdots (*)$$

$$\text{Res}(g; z_2) = \lim_{z \to z_2} \frac{1}{z + \alpha + \sqrt{\alpha^2-1}} = \frac{1}{2\sqrt{\alpha^2-1}}$$

$$\text{So, } \oint_C g(z)\,dz = \frac{\pi i}{\sqrt{\alpha^2-1}}$$

$$\therefore I' = \frac{2}{b} \cdot \frac{\pi i}{\sqrt{\alpha^2-1}} = \frac{2\pi}{\sqrt{a^2-b^2}}$$

$\therefore$ Final Ans

$$I = \frac{2\pi a}{b^2} - \left(\frac{a^2-b^2}{b^2}\right) \frac{2\pi}{\sqrt{a^2-b^2}}$$

$$= \boxed{\frac{2\pi}{b^2}\left(a - \sqrt{a^2-b^2}\right)} \quad D$$

∎