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HW_6

Q1) $$I = P\int_{-\infty}^{\infty} \frac{dw'}{(w'-w)\left[(w'-w_0)^2+a^2\right]}$$

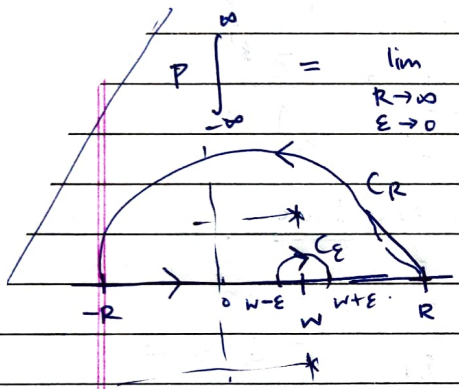
By definition,

since,

$$f(x) := \frac{1}{(x-w)\left[(x-w_0)^2+a^2\right]}$$

is f is continuous (has ∞ singularity) on Real x-axis at $x = w$.

$$P\int_{-\infty}^{\infty} = \lim_{\substack{R\to\infty \\ \varepsilon\to 0}} \left[\int_{-R}^{w-\varepsilon} f(x)\,dx + \int_{w+\varepsilon}^{R} f(x)\,dx \right] = P \text{ (say)}.$$



Now, consider the contour,

$$C = [-R, w-\varepsilon] \oplus C_\varepsilon \oplus [w+\varepsilon, R] \oplus C_R$$

$$\oint_C \frac{dz}{(z-w)\left[(z-w_0)^2+a^2\right]} = P \oint_C \frac{dz}{(z-w)(z-z_0)(z-z_1)}$$

Has a pole inside C.

WLOG, $a > 0$.

$$z - w_0 = \pm ia$$

$$z = w_0 \pm ia$$

$$\boxed{z_0 \equiv w_0 + ia}$$

↓

1st order pole.

$$z_1 = w_0 - ia$$

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$\therefore$ LHS evaluates to $\rightarrow$ (Res due Theorem)

$$\oint_C f(z)\,dz = 2\pi i \operatorname{Res} f(z)\Big|_{z=z_0}$$

$$= 2\pi i \left(\lim_{z \to w_0 + ia} (z - z_0) \frac{1}{(z-w)(z-z_0)(z-\bar{z}_0)} \right)$$

$$= 2\pi i \left[\frac{1}{(w_0 + ia - w)(w_0 + ia - w_0 - ia)} \right]$$

$$= \frac{\pi}{a} \cdot \frac{1}{(w_0 - w + ia)}$$

Now, at $\lim_{\substack{R \to \infty \\ \varepsilon \to 0}}$

RHS evaluates to

$$P + \lim_{\varepsilon \to 0} \int_{C_\varepsilon} \frac{dz}{(z-w)\left[(z-w_0)^2 + a^2\right]} + \lim_{R \to \infty} \int_{C_R} \frac{dz}{(z-w)\left[(z-w_0)^2 + a^2\right]}$$

subst. $z = w + \varepsilon e^{i\theta}$, $0 \le \theta \le \pi$

$dz = \varepsilon i e^{i\theta} d\theta$

$$-\lim_{\varepsilon \to 0} \int_0^\pi \frac{\varepsilon i e^{i\theta} d\theta}{\varepsilon e^{i\theta}\left[(w - w_0 + \varepsilon e^{i\theta})^2 + a^2\right]} = \frac{-i\pi}{(w - w_0)^2 + a^2}$$

$\therefore \left| \frac{z}{(z-w)\left[(z-w_0)^2 + a^2\right]} \right| \to 0$ as $|z| \to \infty$

since $e^{ibz} \to 0$

By Jordan's lemma.

$$\therefore P = \frac{\pi}{a} \frac{(W_0 - W) + ia}{(W_0 - W)^2 + a^2} + \frac{\pi}{a} \frac{ia}{(W_0 - W)^2 + a^2}$$

$$= -\frac{\pi}{a} \left[\frac{(W_0 - W) - ia + ia}{(W_0 - W)^2 + a^2} \right]$$

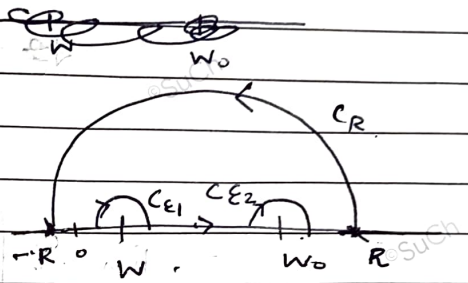
$$\boxed{P = \frac{\pi}{a} \frac{(W_0 - W)}{(W_0 - W)^2 + a^2}}$$

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Case 2: $b \cdot a = 0$.

Then, $$I = P\int_{-\infty}^{\infty} \frac{dx}{(x-w)(x-w_0)^2}$$

Consider. WLOG : $w < w_0$



$$\oint_C f(z)\,dz.$$

$$= \oint_C \frac{dz}{(z-w)(z-w_0)^2}$$

↓

It has 2 poles @

$z = w \rightarrow$ order $= 1$.

$z = w_0 \rightarrow$ order $= 2$.

$$C = [-R, w-\varepsilon_1] \oplus [w+\varepsilon_1, w_0-\varepsilon_2] \oplus [w_0+\varepsilon_2, R] \oplus C_{\varepsilon_1} \oplus C_{\varepsilon_2} \oplus C_R$$

Clearly:

$$P\int_{-\infty}^{\infty} \frac{dx}{(x-w)(x-w_0)^2} = \lim_{\substack{R\to\infty \\ \varepsilon_1\to 0^+ \\ \varepsilon_2\to 0^+}} \left[\int_{-R}^{w-\varepsilon_1} + \int_{w+\varepsilon_1}^{w_0-\varepsilon_2} + \int_{w_0+\varepsilon_2}^{R} f(x)\,dx \right]$$

$\therefore$

$$\oint_C f(z)\,dz = \lim_{R\to\infty} \int_{C_R} f(z)\,dz + P\int_{C_{\varepsilon_1}} f(z)\,dz + \int_{C_{\varepsilon_2}} f(z)\,dz$$

$\varepsilon_1 \to 0^+$; $\varepsilon_2 \to 0^-$

$= 0$ as $f(z)$ is analytic within C.

$\to 0$ by Jordan's Lemma!

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$$-P\int = \lim_{\varepsilon_1 \to 0^+} \int_{C_{\varepsilon_1}} \frac{dz}{(z-w)(z-w_0)^2} + \lim_{\varepsilon_2 \to 0^+} \int_{C_{\varepsilon_1}} \frac{dz}{(z-w)(z-w_0)^2}$$

curve

$z = \varepsilon_1 e^{i\theta} + w$ $\qquad$ $z = \varepsilon_2 e^{i\theta} + w_0$

$dz = i\varepsilon_1 e^{i\theta} d\theta$ $\qquad$ $dz = i\varepsilon_2 e^{i\theta} d\theta$

$$= \lim_{\varepsilon_1 \to 0^+} i \int_{\pi}^{0} \frac{\varepsilon_1 e^{i\theta}\, d\theta}{\varepsilon_1 e^{i\theta} \left(\varepsilon_1 e^{i\theta} + w - w_0\right)^2} \to i$$

$$+ \lim_{\varepsilon_2 \to 0^+} \int_{\pi}^{0} \frac{i\varepsilon_2 e^{i\theta}\, d\theta}{\varepsilon_2^2 e^{2i\theta} \left(\varepsilon_2 e^{i\theta} + w_0 - w\right)}$$

Clearly:

2nd term blows up.

hence, principle value ~~DNE~~ does NOT exist.

Pg-2

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Q2) $\frac{d^2 f(x)}{dx^2} = g(x) \qquad 0 \le x \le 1$

$f(0) = 0$

$f(1) = 0$

A) Consider the Green's Function be, $G(x, x')$

Then, by defⁿ we have,

$$\frac{d^2}{dx^2} G(x, x') = \delta(x - x')$$

T.F $G(x, x') = ?$

~~G should~~

G should also satisfy the same boundary condition, as the solution.

$G(x=0, x') = 0$ $\qquad$ $G(x=1, x') = 0$

And for

$x \neq x'$

$$\frac{\partial^2 G(x, x')}{\partial x^2} = 0 \quad \cdots (*)$$

We have the solution of the differential equation, to be of this form

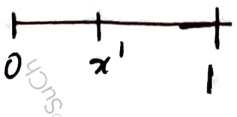
$$f(x) = c_1 y_1(x) + c_2 y_2(x) + \int_0^1 G(x, x') g(x) dx'$$

First we find the homogenous eqⁿ solution

$$\frac{d^2 f_h}{dx^2} = 0 \Rightarrow \frac{df_h}{dx} = c_1 \Rightarrow f_h(x) = c_1 x + c_2$$

From eq (*), we deduce,

$$G(x, x') = \begin{cases} Ax + B & x > x' \\ Cx + D & x < x' \end{cases}$$



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- By property
- $G(x,x')$ should be continuous at $x = x'$

$$\therefore \lim_{x \to x'} G_1(x,x') = \lim_{x \to x'} G_2(x,x')$$

$$\boxed{Ax' + B = Cx' + D} \quad \text{---} (*_1)$$

Note:

$$\mathcal{L} \left(\equiv \frac{d}{dx}\left(p(x) \frac{d}{dx} \right) + q(x) \right) \qquad \left(\mathcal{L} f = g \right)$$

where,

Here, $p(x) = 1$, $q(x) = 0$:

Further, Jump conditions

- $$\lim_{\varepsilon \to 0} \left[\left. \frac{dG}{dx} \right|_{x'+\varepsilon} - \left. \frac{dG}{dx} \right|_{x'-\varepsilon} \right] = \frac{1}{p(x')}$$

$$\boxed{A - C = \frac{1}{p(x')} = 1} \quad \text{---} (*_2)$$

- Boundary Values:

$$G(x=0, x') = 0 = G(x=1, x')$$

$$\boxed{D = 0} \quad \text{---} (*_3)$$

$$\boxed{A + B = 0}$$

using eq $(*_1)$, $(*_2)$, $(*_3)$

$$Ax' - A = Cx'$$

$$A = \frac{Cx'}{x'-1}$$

$$Cx' \left[\frac{x'}{x'-1} - 1 \right] = 1$$

$$C \left[\frac{x' - x' + 1}{x' + 1} \right] = 1$$

$$\boxed{C = x' - 1}$$

$$\therefore \boxed{A = -x'}$$

$$B = -x'$$

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∴ We have, Green's function,

$$G(x,x') = \begin{cases} x x' - x' & x > x' \\ x(x'-1) & x < x' \end{cases}$$

↑ obs: it's not dependent on Forcing Term.

∴ Particular solution,

$$y_p = \int_0^1 G(x,x')\, g(x)\, dx' \equiv f(x)$$

∴ The complete solution is

$$f(x) = c_1 x + c_2 + \int_{x'}^{1} x'(x-1) g(x') dx' + \int_0^{x'} x(x'-1) g(x') dx'$$

Ans

Ans

(3) $L_x = \frac{d^2}{dt^2} + 1$

I.V $y(t=0) = 0, \quad y'(t=0) = 0.$

Let Green's function be $G(t,t')$

We know that

$L\, G(t,t') = \delta(t-t')$ ... same initial

Again, G also satisfies the boundary values.

Problem →

$$G(t=0,t') = 0 \quad \cdots (*)$$

$$\frac{\partial G}{\partial t}(t=0,t') = 0$$

And, $\frac{d^2 G}{dt^2} = -G(t,t')$ For $t \neq t'$

$\ddot{x} = -\omega^2 x$

$x = A\sin(\omega t) + B\cos(\omega t)$

$\omega = 1$

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$\therefore G$ will have oscillatory solution.

~~$t < t'$~~ ~~$t > t'$~~

$$G(t,t') = \begin{cases} A\sin t + B\cos t & t < t' \\ C\sin t + D\cos t & t > t' \end{cases}$$

Properties of

(i) continuity at t'

$$\boxed{A\sin t' + B\cos t' = C\sin t' + D\cos t'} \quad \cdots (*1)$$

discont.

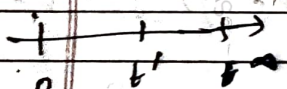
(ii) Jump at deriv.

$$\mathcal{L} \equiv \frac{d}{dt}\left(p(t)\frac{d}{dt}\right) + q(t) = 0$$

Here $p(t) = 1$

$q(t) = 1$

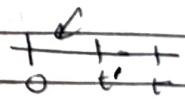
$$\therefore \lim_{\varepsilon \to 0^+} \left[\left.\frac{dG}{dt}\right|_{t'+\varepsilon} - \left.\frac{dG}{dt}\right|_{t'-\varepsilon} \right] = \frac{1}{p(t')} = \frac{1}{1}$$



$$\lim_{\varepsilon \to 0^+} \Big[C\cos(t'+\varepsilon) - D\sin(t'+\varepsilon) - A\cos(t'-\varepsilon) + B\sin(t'-\varepsilon) \Big] = 1$$

$$\Rightarrow \boxed{C\cos(t') - D\sin(t') = A\cos t' - B\sin t' + 1} \quad \cdots (*2)$$

Now, $G(t=0, t') = 0$



$$\Rightarrow \boxed{B = 0}$$

$$\frac{\partial G}{\partial t}(t=0, t') \Rightarrow (A\cos t - B\sin t)\Big|_{t=0} = 0$$

$$\boxed{A = 0}$$

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$\therefore$

$C\cos(t') - D\sin t' = 0 \times (\sin t')$

$C\sin(t') + D\cos t' = 0 \times (\cos t')$

$\Rightarrow C\cos t'\sin t' - D\sin^2 t' = 0$

$C\sin t'\cos t' + D\cos^2 t' = 0$

$-D = \sin t'$

$\therefore \boxed{D = -\sin t'}$

Now $C = \frac{1}{\sin(t')} \cdot -\sin t' \cdot \cos t'$

$\boxed{C = \cos t'}$

For, $t > t'$

$\therefore$

$G(t,t') = \cos t' \sin t - \sin t' \cos t$

Hence, Green's Function for,

$\Delta \equiv \frac{d^2}{dt^2} + 1$ Operator

is

$$G(t,t') = \begin{cases} 0 & t < t' \\ \sin(t-t') & t > t' \end{cases}$$

By, $\tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt$

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④ Green's function

$$\left(\frac{\partial}{\partial t} - \alpha^2 \frac{\partial^2}{\partial x^2}\right) G(x,t;x',t') = \delta(x-x')\,\delta(t-t')$$

Noting translation invariance, we write

$x - x' = X$ | $t - t' = T$ (?) | Sit $G(x,t;x',t') = G(X,T)$

Let, $G(X,T) \overset{FT}{\Longleftrightarrow} \tilde{G}(k,\omega)$

Then we can write.

$$G(X,T) = \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} d\omega\; \tilde{g}(k,\omega)\, e^{i(kX - \omega T)}$$

$$\left(\frac{\partial}{\partial t} - \alpha^2 \frac{\partial^2}{\partial x^2}\right) G(X,T) \equiv \text{LHS}$$

$$= \frac{1}{2\pi} \iint dk\, d\omega\; \tilde{g}(k,\omega) \left[+ \alpha^2 k^2 - i\omega \right] \exp\left(i(kx - \omega T)\right) \quad \cdots (A)$$

↑ can go inside integral operation as t, ω are independent of x, k

$$\frac{\partial}{\partial x} e^{i(kx-\omega T)} = \frac{\partial}{\partial x} e^{i(kx-\omega T)} \cdot ik$$

$$= + ik^2 e^{i(kx-\omega T)}$$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial T}\left(\frac{\partial T}{\partial t}\right) = \frac{\partial}{\partial T}$$

$$\frac{\partial^2}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\right) = \frac{\partial^2}{\partial x^2}$$

Again;

To compare, we find the fourier transform of dirac delta,

$\Rightarrow$ ⓐ $\frac{1}{2\pi}\iint dk\,dw\;\tilde{g}(k,w)\left[\alpha^2k^2-iw\right]\exp\left(i(kx-wT)\right)$ ..... (A)

$= \delta(x)\delta(T)$

We find it's Fourier Transform !!

$$\hat{g}(k,w) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int dx\,dT\;\delta(x)\delta(T)\;e^{-i(kx-wT)}$$

$$= \frac{1}{2\pi}$$

$\therefore$ RHS $\equiv \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int dk\,dw\;e^{i(kx-wT)}$

$$\frac{1}{2\pi} = \tilde{g}(k,w)\left[\alpha^2|k|^2-iw\right]$$

$$\Rightarrow \quad \tilde{g}(k,w) = \frac{1}{2\pi\left(\alpha^2|k|^2-iw\right)}$$

$$\therefore G(x,T) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}dk\int_{-\infty}^{\infty}dw\;\frac{e^{i(kx-wT)}}{\alpha^2|k|^2-iw}$$

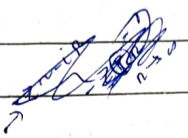
$$= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}dk\;e^{ikx}\int_{-\infty}^{\infty}dw\;\frac{e^{-iwT}}{\alpha^2|k|^2-iw}$$

Look at

$$\int_{-\infty}^{\infty} \frac{dw\, e^{-iwT}}{a^2|k|^2 - iw} \equiv \int_{-\infty}^{\infty} \frac{dx\, e^{-ixT}}{(a^2|k|^2 - ix)}$$

↳ No disc. on real axis

$$= \lim_{R\to\infty} \int_{-R}^{R} \frac{dx\, e^{-ixT}}{(a^2|k|^2 - ix)}$$



$$\oint_C \frac{dz\, e^{-izT}}{(a^2|k|^2 - iz)}$$

singularity at

$$-i\,\frac{a^2|k|^2}{i\theta} = z$$

$$z = -i a^2|k|^2$$

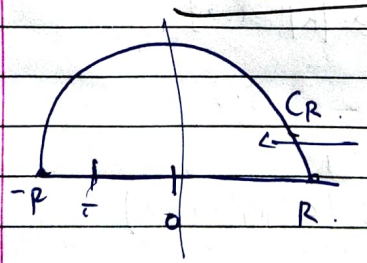
$$\frac{i\left(ia^2|k|^2 - z\right)}{i^2}$$

$$= i\left(-ia^2|k|^2 - z\right)$$

$$= -i\left(z + ia^2|k|^2\right)$$

To use Jordan Lemma,

For $(-T) > 0$, $T < 0$ we choose upper half plane.



$$\therefore \oint_C \frac{dz\, e^{-izT}}{(a^2|k|^2 - iz)} = 0$$

$$= \lim_{R\to\infty} \int_{-R}^{R} + \lim_{R\to\infty} \int_{C_R} f(z)dz$$

From Jordan's Lemma.

Hence

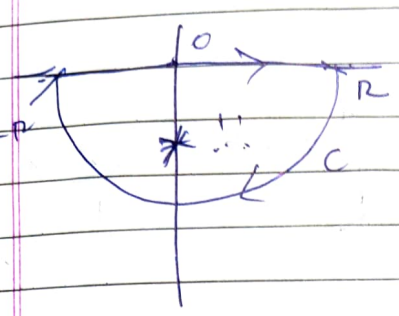
$G(x,T) = 0$ for $T < 0$

$t - t' < 0$

For,

$(-T) < 0$ | We choose Lower half plane.

$T > 0$



Now contour has a pole at

$z_0 = -i a^2 |K|^2$.

→ simple pole.

$$\therefore \oint_C f(z)\,dz = -2\pi i \; \text{Res}\, f(z)\Big|_{z=z_0}$$

$-2\pi i$ ← clockwise contour

$$= -2\pi i \left((z - z_0) \frac{e^{-i z_0 T}}{-i (z - z_0)} \right)$$

$$= 2\pi \exp\left(-T a^2 |K|^2 \right).$$

$$= 2\pi \exp\left(-T a^2 |K|^2 \right). \checkmark$$

$$\text{RHS} = \lim_{R \to \infty} \int_{-R}^{R} + \lim_{R \to \infty} \int_{C_R} f(z)\,dz.$$

By Jordan's Lemma. → 0

So,

$$\int_{-\infty}^{\infty} \frac{dw\, e^{-iwT}}{a^2 |K|^2 - iw} = 2\pi e^{-T a^2 |K|^2}. \quad T > 0.$$

$\checkmark$

$$G(x,T) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} dk\, e^{ikx} (2\pi) e^{-Ta^2k^2}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \exp\left[-T\left[a^2k^2 - \frac{ikx}{T}\right]\right]$$

Completing square.

$$a^2k^2 - \frac{2iaKx}{2aT}$$

$$= a^2k^2 - 2(ak)\left(\frac{ix}{2aT}\right) + \left(\frac{ix}{2aT}\right)^2 + \left(\frac{ix}{2aT}\right)^2 - \left(\frac{ix}{2aT}\right)^2$$

$$= \left(ak - \frac{ix}{2T}\right)^2 + \left(\frac{x}{2aT}\right)^2$$

$$\therefore G(x,T) = \frac{e^{-T\cdot(x/2aT)^2}}{2\pi} \int_{-\infty}^{\infty} dk \exp\left[-T\left(ak - \frac{ix}{2T}\right)^2\right]$$

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Let, $h = ak - \frac{ix}{2T}$

Since, K is very large. a over integral.

$$h \simeq ak$$

$$\frac{dh}{a} = dk.$$

$$G(x,T) = \frac{e^{-T\cdot(x/2a)^2}}{2\pi}\cdot\frac{1}{a}\underbrace{\int_{-\infty}^{\infty} dh\cdot e^{-Th^2}}_{\sqrt{\frac{\pi}{T}}}$$

$$-T\frac{x^2}{4a^2Tx}$$

$$G(x,T) = \frac{e^{-x^2/4a^2T}}{2\pi a}\sqrt{\frac{\pi}{T}}$$

$\therefore$

$$G(x,t;x',t') = \begin{cases} 0 & t < t' \\ \frac{1}{2\pi a}\cdot\sqrt{\frac{\pi}{t-t'}}\, e^{-\frac{(x-x')^2}{4a^2(t-t')}} & t > t' \end{cases}$$

HW:

1) $f(x) - \lambda\int_0^1 dy\, K(x,y) f(y) = 1$

(a) Kernel $K(x,y) = xy$.

The general form is:

$$h(x) f(x) = \lambda\int_0^1 dy\, K(x,y) f(y) + g(x).$$

Here, $h(x) = 1$,

$g(x) = 1$.

$$K(x,y) = xy \equiv \phi_1(x)\psi_1^*(y) \equiv \sum_{j=1}^{n}\phi_j(x)\psi_j^*(y)$$

$\phi_1 = x$, $\psi_1^* = y$

so, since we have a finite sum & a separable kernel.

This is a degenerate kernel, so we hope to get an exact solution.

$$f(x) = \lambda \int_0^1 dy\ xy\ f(y) + 1.$$

Define.

$$A := \int_0^1 dy\ y\ f(y)$$

Put,

$$f(x) = 1 + \lambda A x.$$

Plug it back.

$$A = \int_0^1 dy\ y\ (1 + \lambda A y)$$

$$= \int_0^1 dy\ \left(y + \lambda A y^2\right)$$

$$\frac{3A}{3} = \frac{1}{2} + \frac{\lambda A}{3}$$

$$\frac{A}{3}(3 - \lambda) = \frac{1}{2}$$

$$A = \frac{-3}{2} \cdot \frac{1}{(\lambda - 3)}$$

$$\therefore f(x) = 1 - \frac{3}{2} \frac{\lambda x}{(\lambda - 3)}$$

Hence the solution breaks down for $\boxed{\lambda = 3}$.

(b) $K(x,y) = \sin(\pi x)\sin(\pi y)$

Similarly

$K(x,y) \equiv \phi_1(x)\,\psi_1^*(y)$

$\phi_1 = \sin \pi x$

$\psi_1 = \sin(\pi y)$.

$$f(x) = \lambda \int_0^1 dy \; \sin(\pi x)\cdot \sin(\pi y)\, f(y) \quad +1$$

Define

$$B := \int_0^1 dy \sin(\pi y) f(y)$$

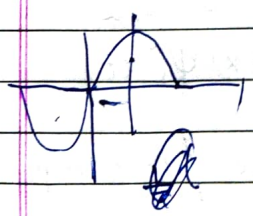
$\therefore$ Put

$$f(x) = 1 + \lambda B \sin(\pi x).$$

Plug it back!

$$B := \int_0^1 dy \left(\sin \pi y \left(1 + \lambda B \sin \pi y\right)\right)$$

$$B := \int_0^1 dy \left(\sin \pi y + \lambda B \sin^2 \pi y \right).$$



$\pi y = h \quad dy = \frac{dh}{\pi}$

$$= \frac{2}{\pi} \int_0^{\pi/2} dh \left(\sin h + \lambda B \sin^2 h\right)$$

$\int \sin = (-\cos)\Big|_0^{\pi/2}$

$= 1$

$$= \frac{2}{\pi} \times 1 + \frac{2}{\pi} \int_0^{\pi/2} \lambda B \sin^2 h \, dh$$

I'

Note: (King Rule)

$I' = \lambda B \times \frac{\pi}{4}$

$$\frac{2B}{2} = \frac{2}{\lambda} + \frac{\lambda}{\lambda} B \cdot \frac{\pi}{2}$$

$$-\frac{2}{\lambda} = \frac{B}{2}(\lambda - 2)$$

$$B = \frac{-4}{\lambda(\lambda-2)}$$

Hence,

$$f(x) = 1 - \frac{4\lambda}{\pi(\lambda-2)} \sin(\pi x)$$

Bad point, $\lambda = 2$

② $O(\lambda^3)$ Iterative solution.

$$y(x) = x + \frac{\lambda}{2} \int_{-1}^{1} (t-x)\, y(t)\, dt$$

Notice here the Kernel.

$K(x,t) = \frac{(t-x)}{2}$ cannot be written in the form.

$$\sum_j \Phi_j(x) \Psi_j^*(t)$$

So, this is Non-degenerate / Non-separable Kernel.

We look for iterative solutions for $y(x)$

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$$f(x) = \underbrace{x}_{g(x)} + \frac{1}{2}\int_{-1}^{1} (t-x)\, f(t)\, dt .$$

(i) Initial zeroth order guess.

$$f_0(x) = g(x) = x \quad \text{(Same)}.$$

We are working under assumption that -

$$x \gg \int_{-1}^{1} (t-x)\, f(t)\, dt .$$

(ii) 1st order Guess : $f_1(x)$

$$f_1(x) = g(x) + \frac{1}{2}\int_{-1}^{1} (t-x)\, f_0(t)\, dt .$$

$f_0(t) = g(t)$!!

$$f_1(x) = x + \frac{1}{2}\int_{-1}^{1} (t-x)\, dt \cdot \int_{-1}^{1} (t^2 - xt)\, dt$$

$$= 2\int_{0}^{1} t^2 = 2 \cdot \frac{1}{3} = \frac{2}{3}$$

$g(x)$: $\frac{1}{2} \cdot \frac{2}{3}$

$$= x + \frac{1}{3}$$

$$f_1(x) = x + \frac{1}{3}$$

(iii) 2nd order Guess : $f_2(x)$.

$$f_2(x) = g(x) + \frac{1}{2}\int_{-1}^{1} (t-x)\, f_1(t)\, dt .$$

Any continuous function in a odd interval can be written as a convergent series of a polynomial !!

$$f_2(x) = x + \frac{1}{2}\int_{-1}^{1} (t-x)\left(t+\frac{1}{3}\right) dt$$

$$= x + \frac{1}{2}\int_{-1}^{1} t^2 - xt + \frac{t}{3} - \frac{x}{3} dt$$

$$= x + 1 \int_{0}^{1} \left(t^2 - \frac{x+1}{3}\right) dt$$

$$= x + 1 \left[\frac{1}{3} - \frac{x+1}{3}\right]$$

$$\boxed{f_2(x) = x + \frac{1}{3} - \frac{x+1^2}{3}}$$

(iii) 3rd order given $f_3(x)$

$$f_3(x) = g(x) + 1\int_{-1}^{1} K(x,t) \cdot f_2(t)\, dt$$

$$= x + \frac{1}{2}\int_{-1}^{1} (t-x)\left[t + \frac{1}{3} - \frac{t+1^2}{3}\right] dt$$

$$= x + \frac{1}{2}\int_{-1}^{1} \left[t^2 + \frac{t}{3} - \frac{t^2+1^2}{3} - xt - \frac{x}{3} + \frac{x(t^2+1^2)}{3}\right] dt$$

$$= x + \lambda \int_0^1 \left[t^2 \left(1 - \frac{t^2}{3} \right) - x \frac{\lambda}{3} \right] dt.$$

$$f_3(x) = x + \lambda \left[\frac{1}{3} \left(1 - \frac{\lambda^2}{3} \right) - \frac{x\lambda}{3} \right].$$

$$f_3(x) = x + \lambda \left[\frac{1}{3} - \frac{\lambda^2}{9} - \frac{x\lambda}{3} \right]$$

$$f_3(x) = x + \frac{\lambda}{3} - \frac{\lambda^3}{9} - x\frac{\lambda^2}{3}.$$