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PH 3103 Assignment (HW-2)

8) $f(\theta,\phi) = a + b\cos\theta + c\sin^2\theta + d\sin\theta\cos\phi + h\sin^2\theta\sin(2\phi)$

Ans We know $Y_l^m(\theta,\phi) = N_{lm} P_l^m(\cos\theta) e^{im\phi}$

where $P_l^m(x)$ → is the associated Legendre' Polynomial

And $x \in [-1,1]$

$$P_l^m(x) := (1-x^2)^{\frac{m}{2}} \frac{d^m}{dx^m} P_l(x) \quad \Big| \; l \in \mathbb{Z}^+, \; m = 0,1,\cdots l$$

(From Rodrigue's formula)

where

$$P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l}\left[(x^2-1)^l\right] \longrightarrow l^{th} \text{ degree polynomial in } x.$$

$$\therefore \quad P_l^m(\cos\theta) = (\sin\theta)^m \frac{d^m}{dx^m} P_l(\cos\theta).$$

N_{lm} is Normalization constant.

So, since $P_l(\cos\theta) \sim (\cos\theta)^l$

$$\therefore \frac{d^m}{dx^m}(\cos\theta) \sim (\cos\theta)^{l-m}.$$

$$\therefore Y_l^m(\theta,\phi) \sim (\sin\theta)^m (\cos\theta)^{l-m} e^{im\phi}$$

We proceed to compare the powers of sin, cos.

$a \longrightarrow l = m = 0 \rightarrow \therefore \sim Y_0^0$

$b\cos\theta \rightarrow m = 0$ & $l - 0 = 1 \rightarrow l = 1$

$c\sin^2\theta \rightarrow m = 2$ & $l - 2 = 0 \Rightarrow l = 2$

$d\sin\theta\cos\phi \rightarrow m = 1$, $l - 1 = 0 \rightarrow l = 1$

~~$2h\sin^3\theta\cos\theta \rightarrow m = 3$, $l - 3 = 1 \rightarrow l = 4$~~

$h\sin^2\theta\sin(2\phi) \rightarrow m = 2$, $l - 2 = 0 \rightarrow l = 2$.

$C_{lm} Y_l^m$

$\therefore$ Let observation $f(\theta,\phi) = \underbrace{C_{00} Y_0^0}_{\text{constant}} + \underbrace{C_{1,0} Y_1^0}_{\cos\theta} + \underbrace{C_{2,2} Y_2^2 + C_{2,-2} Y_2^{-2}}_{\sin^2\theta\,(\sin(2\phi))} + C_{2,0} Y_2^0 + \underbrace{C_{1,1} Y_1^1 + C_{1,-1} Y_1^{-1}}_{\sin\theta\cos\phi}$

parameters $K_1 = \sqrt{\frac{3}{8\pi}}$, $K_2 = 3\sqrt{\frac{5}{96\pi}}$

(2)

$b\cos\theta = C_{1,0}\, K_1\sqrt{2}\cos\theta \rightarrow \boxed{C_{1,0} = \frac{b}{K_1\sqrt{2}}}$

$h\sin^2\theta\sin 2\phi = C_{2,2}\,Y_2^{\,2} + C_{2,-2}\,Y_2^{\,-2}$

$= C_{2,2}\,K_2\sin^2\theta\, e^{2i\phi} + C_{2,-2}\,K_2\sin^2\theta\, e^{-2i\phi}$

$= K_2\sin^2\theta\left[C_{2,2}(\cos 2\phi + i\sin(2\phi)) + C_{2,-2}(\cos(2\phi) - i\sin(2\phi))\right]$

$h = K_2 C_{2,2}\, i + K_2 C_{2,-2}(-i)$

$0 = K_2 C_{2,2} + K_2 C_{2,-2} \rightarrow \boxed{C_{2,2} = -C_{2,-2}}$

$i\frac{h}{K_2} = +C_2\, C_{2,-2} \rightarrow \boxed{C_{2,-2} = \frac{i}{2}\frac{h}{K_2}}$

$c\sin^2\theta + a = C_{0,0}\,Y_0^0 + C_{2,0}\,Y_2^0$

$\Rightarrow (c+a) - c(\cos^2\theta) = C_{0,0}\frac{1}{\sqrt{4\pi}} + C_{2,0}\sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$

$-c = C_{2,0}\frac{3}{2}\sqrt{\frac{5}{4\pi}} \rightarrow \boxed{C_{2,0} = -\frac{2}{3}\sqrt{\frac{4\pi}{5}}}$

$a + c = \frac{C_{0,0}}{\sqrt{4\pi}} - C_{2,0}\frac{1}{2}\sqrt{\frac{5}{4\pi}}$

$= \frac{C_{0,0}}{\sqrt{4\pi}} + \frac{1}{2}\left(\frac{2}{3}\right)\sqrt{\frac{4\pi}{5}}\sqrt{\frac{5}{4\pi}}$

$\rightarrow \boxed{C_{0,0} = \left(a + c - \frac{1}{3}\right)\sqrt{4\pi}}$

$d\sin\theta\cos\phi = C_{1,1}\,Y_1^{\,1} + C_{1,-1}\,Y_1^{\,-1}$

$= C_{1,1}\cdot(-K_1)\sin\theta\, e^{i\phi} + C_{1,-1}\,K_1\sin\theta\, e^{-i\phi}$

$= K_1\sin\theta\left[(-C_{1,1} + C_{1,-1})\cos\phi + (-iC_{1,1} - iC_{1,-1})\sin\phi\right]$

comparing & solving. $\boxed{C_{1,-1} = \frac{d}{2K_1} = -C_{1,1}}$.

Ans $K_1 = \sqrt{\frac{3}{8\pi}}$, $K_2 = 3\sqrt{\frac{5}{96\pi}}$

$f(\theta,\phi) = \left(a + c - \frac{1}{3}\right)\sqrt{4\pi}\,Y_0^0 + \frac{b}{K_1\sqrt{2}}Y_1^0 + \left(\frac{d}{2K_1}\right)\left(-Y_1^{\,1} + Y_1^{\,-1}\right) - \frac{2}{3}\sqrt{\frac{4\pi}{5}}Y_2^0$

$+ \frac{ih}{2K_2}\left(-Y_2^{\,2} + Y_2^{\,-2}\right)$

(3) $2J_n' = J_{n-1} - J_{n+1}$

Generating Function

$g(x,t) = e^{x/2\,(t-\frac{1}{t})} = \sum_n J_n(x)\, t^n$.

We have

$\Rightarrow \frac{\partial g}{\partial x} = e^{x/2\,(t-\frac{1}{t})} \left(t - \frac{1}{t}\right) \cdot \frac{1}{2} = \sum_n J_n'(x)\, t^n$.

$\Rightarrow \frac{g(x,t)\,(t-t^{-1})}{2} = \sum_n J_n' t^n$.

$\Rightarrow (t-t^{-1}) \sum_n J_n(x)\, t^n = 2 \sum_n J_n'(x)\, t^n$

$\Rightarrow \sum_K (J_{K-1}(x) - J_{K+1}(x))\, t^K = \sum_K 2 J_K'(x)\, t^K$

(changing Indices)

$\Rightarrow J_{n-1}(x) - J_{n+1}(x) = 2 J_n'(x) \quad \forall\, n \in \mathbb{Z}^+$. □

(2) $J_{1/2}(x) = ?$

We have, $J_\nu(x) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\,\Gamma(\nu+j+1)} \left(\frac{x}{2}\right)^{\nu+2j}$ $\nu \in \mathbb{R}^*$, ~~$\nu \in \mathbb{Z}$~~.

$\therefore J_{1/2}(x) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\,\Gamma(j+\frac{3}{2})} \left(\frac{x}{2}\right)^{\frac{1}{2}+2j}$

Now, $\Gamma(s) = \int_0^\infty t^{s-1} e^{-t}\, dt$ $\therefore \Gamma(j+\frac{3}{2}) = \int_0^\infty t^{j+1/2} e^{-t}\, dt$

By defⁿ,

$= \left(\frac{t^{j+1/2}}{-1} e^{-t} \right|_0^\infty + \left(j+\frac{1}{2}\right) \int_0^\infty t^{j-1/2} e^{-t}\, dt$

$= \left(j+\frac{1}{2}\right) \Gamma\left(j+\frac{1}{2}\right) = \left(j+\frac{1}{2}\right)\left(j-\frac{1}{2}\right)\left(j-\frac{3}{2}\right)\cdots\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)$

$= \frac{(2j+1)\,2j\,(2j-1)\,(2j-2)\,(2j-3)\cdots 3!\,2!\,1}{(2^{j+1})\,(2j)\,(2j-2)\cdots 2!}\,\Gamma\left(\frac{1}{2}\right)$

$= \frac{(2j+1)!}{2^{2j+1}\,j!}\sqrt{\pi}$ $\left(\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right)$

So,

$J_{1/2}(x) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \frac{2^{2j+1}\, j!}{(2j+1)!\sqrt{\pi}} \left(\frac{x}{2}\right)^{2j+1/2}$

$= \left(\frac{x}{2\pi}\right)^{1/2} \frac{1}{x} \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)!}\, x^{2j+1}$

$= \left(\frac{x}{2\pi}\right)^{1/2} \frac{1}{x} \sin(x)$.

Series Convergence:

$$E_j = \frac{(-1)^j}{(2j+1)!} x^{2j+1}$$

By Ratio Test,

$$\left|\frac{c_{j+1}}{c_j}\right| = \left| \frac{(-1)^{j+1} x^{2j+3}}{(2j+3)(2j+2)(2j+1)!} \cdot \frac{(2j+1)!}{(-1)^j x^{2j+1}} \right|$$

$$= \left| \frac{x^2}{(2j+3)(2j+2)} \right| \longrightarrow 0 \quad \forall x \in \mathbb{R}$$

as $j \to \infty$

Hence the series converges, and we have

$$\boxed{J_{\nu_2}(x) = \sqrt{\frac{2}{\pi x}} \sin x}$$ in closed form.

Q.1) c) P.T $\int_0^{\pi/2} J_0(x\cos\theta)\sin\theta\, d\theta = \frac{\sin x}{x}$

$$J_\nu(x) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\,(j+\nu)!} \left(\frac{x}{2}\right)^{\nu+2j} \rightarrow J_0(x) = \sum_{j=0}^{\infty} \frac{(-1)^j}{(j!)^2} \left(\frac{x}{2}\right)^{2j}$$

$$\therefore \int_0^{\pi/2} J_0(x\cos\theta)\cos\theta\, d\theta = \int_0^{\pi/2} \sum_{j=0}^{\infty} \frac{(-1)^j}{(j!)^2} \left(\frac{x}{2}\right)^{2j} (\cos\theta)^{2j+1}\, d\theta$$

Now given

$$\int_0^{\pi/2} (\cos\theta)^{2s+1}\, d\theta = \frac{\sqrt{\pi}\, s!\, 2^s \cdot \frac{1}{2}}{(s+\frac{1}{2})(s-\frac{1}{2})(s-\frac{3}{2})\cdots(\frac{1}{2})\sqrt{\pi}} = \frac{1}{2}\frac{2^{s+1}\, s!\,(2s)(2s-2)\cdots 2}{(2s+1)(2s-1)(2s-3)\cdots 1}$$

$$\therefore I = \frac{1}{x} \sum_{j=0}^{\infty} \frac{(-1)^j}{(j!)^2} \left(\frac{x}{2}\right)^{2j+1} \frac{1}{2^{2j+1}} \frac{(j!)^2}{(2j+1)!} = \frac{1}{2}\frac{2^{2s+1}(s!)^2}{(2s+1)!}$$

$$= \frac{1}{x}\left(\sum_{j=0}^{\infty} \frac{(-1)^j x^{2j+1}}{(2j+1)!}\right) = \frac{\sin x}{x}$$ proved □

(b) T.P $e^{ix\cos\theta} = \sum_{m=-\infty}^{\infty} i^m J_m(x) e^{im\theta}$

we know that-

$z \in \mathbb{C}$ $e^{\frac{x}{2}\left(z-\frac{1}{z}\right)} = J_0 + zJ_1 + z^2 J_2 + \cdots z^{-1}J_{-1} + z^{-2}J_{-2} + z^{-3}J_{-3} + \cdots$

Subst. $z = e^{i\theta}$

$$\therefore e^{ix\sin\theta} = e^{\frac{x}{2}(e^{ix} - e^{-i\theta})} = J_0 + J_1 e^{i\theta} + J_2 e^{2i\theta} + \cdots + J_{-1} e^{-i\theta} + J_{-2} e^{-2i\theta} + J_{-3} e^{-3i\theta} + \cdots$$

$$= \sum_{m=-\infty}^{\infty} J_m e^{im\theta}$$

1)b)

change $\theta \to \theta + \pi/2$

$\therefore \sin\theta \to \sin(\theta + \pi/2) = \cos\theta$ $\therefore$ LHS $= e^{ix\cos\theta}$

RHS

$$\sum_{m=-\infty}^{\infty} J_m e^{im(\theta+\pi/2)} = \sum_m J_m(x)\, e^{im\theta}\left[\cos\left(\frac{m\pi}{2}\right) + i\sin\left(\frac{m\pi}{2}\right)\right]$$

look at

$$\left[\cos\left(\frac{m\pi}{2}\right) + i\sin\left(\frac{m\pi}{2}\right)\right]$$

$= i^m$

By induction!

$m=-1 \to -i = i^{-1}$

$m=0 \to 1 = i^0$

$m=1 \to i = i^1$

$m=2 \to -1 = i^2$

$m=3 \to -i = i^3$

$m=4 \to 1 = i^4$

Thus $e^{ix\cos\theta} = \sum_{m=-\infty}^{\infty} i^m J_m(x)\, e^{im\theta}$ □

Q1) a) $J_n(x+y) = \sum_{m=-\infty}^{\infty} J_{n-m}(x)\, J_m(y)$

From generating functions we have

$$\sum_{n=-\infty}^{\infty} J_n(x+y)\, t^n = e^{\frac{x+y}{2}(t-\frac{1}{t})} = e^{\frac{x}{2}(t-\frac{1}{t})} \cdot e^{\frac{y}{2}(t-\frac{1}{t})}$$

$$= \left(\sum_{k=-\infty}^{\infty} J_k(x)\, t^k\right)\left(\sum_{m=-\infty}^{\infty} J_m(y)\, t^m\right)$$

$$= \sum_{k,m=-\infty}^{\infty} J_k(x)\, J_m(y)\, t^{k+m}$$

subst:

$k+m = n$

$\therefore k = n-m$

$$= \sum_{n=-\infty}^{\infty}\left(\sum_m J_{n-m}(x)\, J_m(y)\right) t^n$$

$\therefore$ upon comparing coefficients

$$J_n(x+y) = \sum_{m=-\infty}^{\infty} J_{n-m}(x)\, J_m(y) \quad \forall\, n$$

□

Q4) Tip $j_1(x) = -\frac{\cos x}{x} + \frac{\sin x}{x^2}$

By definition we have,
spherical bessel function.

$$y = j_n(x) = \sqrt{\frac{\pi}{2x}}\, J_{n+y_2}(x)$$

∴ For $n = 1$

$$j_1(x) = \sqrt{\frac{\pi}{2x}}\, J_{3/2}(x)$$

$$j_0(x) = \sqrt{\frac{\pi}{2x}} \cdot J_{y_2}(x) = \sqrt{\frac{\pi}{2x}} \cdot \sqrt{\frac{2}{\pi x}} \sin x = \frac{\sin x}{x}$$ (we calculated in Q. 2).

Now, from Recurrence Relation we have,

$$\frac{d}{dx}\left[x^{-n} j_n(x)\right] = -x^{-n} j_{n+1}(x)$$

put, $n = 0$

$$\therefore j_1(x) = -\frac{d}{dx}[j_0] = -\left(-\frac{\sin x}{x^2} + \frac{\cos x}{x}\right)$$

$$= \frac{\sin x}{x^2} - \frac{\cos x}{x}.$$

□.

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(5) We Know,

$$\sum_{l=0}^{\infty} z^l P_l(x) = (1 - 2xz + z^2)^{-1/2}.$$

(There's a typo in the question).

Now, given, $P_l(0) = \frac{(-1)^{l/2}}{(l/2)!} \frac{1}{2} \cdot \frac{3}{2} \cdot \cdots \frac{l-1}{2}$ for l is even,

we instead prove this for $P_{2n}(0)$ ~~= 0~~,

& $P_{2n+1}(0) = 0$ for $n = 0, 1, 2, \ldots$

(5) we have,

$$\sum_{n=0}^{\infty} z^{2n} P_{2n}(x) = (1 - 2xz + z^2)^{-1/2}.$$

on putting $x = 0$.

$$\sum z^{2n} P_{2n}(0) := (1 + z^2)^{-1/2}$$

Expanding.

Q4) Tip $j_1(x) = -\frac{\cos x}{x} + \frac{\sin x}{x^2}$

By definition we have,
spherical bessel function.

$$y = j_n(x) = \sqrt{\frac{\pi}{2x}}\, J_{n+\frac{1}{2}}(x)$$

∴ For $n = 1$

$$j_1(x) = \sqrt{\frac{\pi}{2x}}\, J_{3/2}(x)$$

$$j_0(x) = \sqrt{\frac{\pi}{2x}} \cdot J_{1/2}(x) = \sqrt{\frac{\pi}{2x}} \cdot \sqrt{\frac{2}{\pi x}} \sin x = \frac{\sin x}{x}$$ (we calculated in Q. 2).

Now, from Recurrence Relation we have,

$$\frac{d}{dx}\left[x^{-n} j_n(x)\right] = -x^{-n} j_{n+1}(x)$$

put, $n = 0$

$$\therefore j_1(x) = -\frac{d}{dx}[j_0] = -\left(-\frac{\sin x}{x^2} + \frac{\cos x}{x}\right)$$

$$= \frac{\sin x}{x^2} - \frac{\cos x}{x}.$$

□

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(5) We know, $\sum_{l=0}^{\infty} z^l P_l(x) = (1 - 2xz + z^2)^{-1/2}$.

(There's a typo in the question).

Here, given, $P_l(0) = \frac{(-1)^{l/2}}{(l/2)!} \cdot \frac{1}{2} \cdot \frac{3}{2} \cdot \cdots \frac{l-1}{2}$ for l is even,

we instead prove this for $P_{2n}(0)$ & $P_{2n+1}(0) = 0$ for $n = 0, 1, 2, \ldots$

(6) we have,

$$\sum_{n=0}^{\infty} z^{2n} P_{2n}(x) = (1 - 2xz + z^2)^{-1/2}.$$

on putting $x = 0$.

$$\sum z^{2n} P_{2n}(x) = (1 + z^2)^{-1/2}$$

Expanding.

∴ ~~For even~~

$$= 1 + \left(-\frac{1}{2}\right)z^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(z^2)^2}{\lfloor 2} + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)(z^2)^3}{3!}$$

$$+ \cdots + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdots\left(-\frac{1}{2}-n+1\right)}{n!}(z^2)^n.$$

so for even powers of z,

i.e for $l = 2n$.

we can easily see.

$$P_{2n}(0) = \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\left(-\frac{1}{2}-n+1\right)}{n!}$$

$$= (-1)^n \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n\, n!}$$

$$\Rightarrow P_{2n}(0) = (-1)^n \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdot 8\cdots 2n}.$$

$$(\Rightarrow) \boxed{P_l(0) = \frac{(-1)^{l/2}}{(l/2)!}\,\frac{1}{2}\cdot\frac{3}{2}\cdots\frac{l-1}{2}.}$$

if ~~l~~ l is even

0

And, clearly odd powers of z do not appear at all. z^{2n+1}

ie $P_{2n+1}(0) = 0$. $\leftrightarrow$ $P_l(0) = 0$ for l - odd

D'

7) Expand in terms of P_n's

$$f(x) = \begin{cases} -1 & -1 \le x < 0 \\ +1 & 0 < x \le 1 \end{cases}$$

Am

$$f(x) = \sum_{n=1,3,5,7,\ldots}^{\infty} \left[(2n+1)\int_0^1 P_n(x)\,dx\right] P_n(x)$$

Am.

Am

Let $f(x) = \sum_{n=0}^{\infty} a_n P_n(x)$.

$$\Rightarrow \int_{-1}^{1} f(x) P_m(x) = \int_{-1}^{1} \sum_{n=0}^{\infty} a_n P_n(x) P_m(x)\,dx$$

$$\Rightarrow \int_{-1}^{1} f(x) P_m(x)\,dx = \sum_{n=0}^{\infty} a_n \int_{-1}^{1} P_n(x) P_m(x)\,dx.$$

From orthogonality relations derived in class, we have

$$= \sum_{n=0}^{\infty} \frac{2a_n}{2n+1} \delta_{mn}.$$

$$\int_{-1}^{1} f(x) P_m(x)\,dx = \frac{2}{2m+1} a_m.$$

$$\therefore a_m \frac{2}{2m+1} = \int_0^1 P_m(x)\,dx - \int_{-1}^{0} P_m(x)\,dx$$

$$= \int_0^1 P_m(x)\,dx + \int_1^0 P_m(-y)\,dy \quad \text{(subst. } x = -y\text{)}$$

$$= \int_0^1 P_m(x)\,dx + \int_1^0 (-1)^m P_m(y)\,dy \quad \text{(by property)}$$

For m = even,

$$\int_1^0 (-1)^m P_m(y) = -\int_0^1 P_m(y)\,dy \quad \therefore a_m = 0 \ \forall\ m \neq \text{even}.$$

For m = odd, $\int_1^0 (-1)^m P_m(y) = \int_0^1 P_m(y)\,dy$

$$\therefore \frac{2}{2m+1} a_m = \int_0^1 P_m(x)\,dx \quad \Rightarrow a_m = (2m+1)\int_0^1 P_m(x)\,dx$$

$m = 1, 3, 5, \ldots$

~~$m = 2n+1,\ n = 0, 1, 2, 3, \ldots$~~

Q 9) $\Psi(r=a, \theta, \phi) = 0$

E = Energy of particle

(a) $\Psi = R(r)\,\Theta(\theta)\,\Phi(\phi)$

Ans As no information is given, regarding potential, we consider it to be a ~~free particle~~, i.e $V = 0$ (inside the well).

$$\hat{H}\Psi = E\Psi$$

$$\Rightarrow -\frac{\hbar^2}{2m}\nabla^2\Psi = E\Psi$$

$$\boxed{\begin{aligned} V &= 0, \quad r \le a \\ V &= \infty, \quad r > a \end{aligned}}$$

in spherical coordinates.

$$\frac{1}{r^2\sin\theta}\left[\sin\theta\frac{\partial}{\partial r}\left(r^2\frac{\partial\Psi}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Psi}{\partial\theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2\Psi}{\partial\phi^2}\right] = \frac{-2mE}{\hbar^2}\Psi = -K^2\Psi$$

we note $K^2 = \frac{2mE}{\hbar^2}$.

we plug Ψ & divide by $R\,\Theta\,\Phi$.

$$\Rightarrow \frac{1}{r^2\sin\theta}\left[\frac{1}{R}\sin\theta\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{1}{\Phi}\frac{1}{\sin\theta}\frac{d^2\Phi}{d\phi^2}\right] = -K^2$$

$\Rightarrow$ Let

$$\frac{-1}{\Phi}\frac{d^2\Phi}{d\phi^2} = \frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left[\sin\theta\frac{d\Theta}{d\theta}\right] + \frac{\sin^2\theta}{R}\frac{d}{dr}\left[r^2\frac{dR}{dr}\right] + K^2r^2\sin^2\theta = \alpha.$$

LHS $\rightarrow$ explicit fn of Φ, RHS $\rightarrow$ " " of θ, R, r,] both can be equal iff they are equal to some constant.

$$\Rightarrow \boxed{\frac{d^2\Phi}{d\phi^2} + \alpha\Phi = 0}$$ ODE for $\Phi(\phi)$

Now,

$$K^2r^2\sin^2\theta + \frac{\sin^2\theta}{R}\frac{d}{dr}\left[r^2\frac{dR}{dr}\right] + \frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left[\sin\theta\frac{d\Theta}{d\theta}\right] = \alpha$$

Now

$$\therefore \underbrace{k^2 r^2 + \frac{1}{R}\frac{d}{dr}\left[r^2 \frac{dR}{dr}\right]}_{\text{Pure Radial Part}} = \underbrace{\frac{\alpha}{\sin^2\theta} - \frac{1}{\Theta \sin\theta}\frac{d}{d\theta}\left[\sin\theta \frac{d\Theta}{d\theta}\right]}_{\text{Pure } \theta \text{ part}} = \beta \quad (\text{Under same logic})$$

$$\therefore \boxed{\frac{2r}{R}\frac{dR}{dr} + \frac{r^2}{R}\frac{d^2R}{dr^2} + k^2 r^2 = \beta} \quad \text{ODE for } R(r)$$

And $$\frac{\alpha}{\sin^2\theta} - \frac{1}{\Theta \sin\theta}\left[\cos\theta \frac{d\Theta}{d\theta} + \sin\theta \frac{d^2\Theta}{d\theta^2}\right] = \beta.$$

$$\Rightarrow \frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} + \frac{1}{\Theta}\cot\theta \frac{d\Theta}{d\theta} - \frac{\alpha}{\sin^2\theta} = -\beta.$$

$$\Rightarrow \boxed{\frac{d^2\Theta}{d\theta^2} + \cot\theta \frac{d\Theta}{d\theta} - \frac{\Theta\alpha}{\sin^2\theta} + \beta\Theta = 0}$$

ODE for $\Theta(\theta)$.

Azimuthal part;

$$\frac{d^2\Phi}{d\phi^2} = -\alpha\Phi \Rightarrow \Phi = C e^{\pm i\sqrt{\alpha}\phi}$$

→ (Both should have same coeff as clockwise, Anticlockwise motion depends on convention).

Now since particle is in a sphere, we require

$$\Phi(\phi) = \Phi(\phi + 2\pi) \quad \forall \phi$$

$$\therefore C e^{\pm i\sqrt{\alpha}\phi} = C e^{\pm i\sqrt{\alpha}\phi} \cdot e^{\pm i\sqrt{\alpha}\, 2\pi}$$

$$\Rightarrow e^{\pm i\sqrt{\alpha}\, 2\pi} = 1.$$

This only possible if $\sqrt{\alpha} = m$ (say) $m \in \mathbb{Z}$.

$$\rightarrow \boxed{\alpha = m^2}$$

$$\boxed{\Phi(\phi) = \frac{1}{\sqrt{2\pi}} e^{\pm im\phi}}$$

→ After Normalization.

For Angular part, Let $\Theta(\theta) = y(x)$, $x = \cos\theta$.

$$\therefore \frac{d\Theta}{d\theta} = \frac{dy}{d\theta} = \frac{dy}{dx}\cdot\frac{dx}{d\theta} = -\sin\theta \frac{dy}{dx}$$

$$\therefore \frac{d^2\Theta}{d\theta^2} = x\frac{dy}{dx} + (1-x^2)\frac{d^2y}{dx^2}.$$

Substituting

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \left[\beta - \frac{\alpha}{1-x^2}\right]y$$

$$\Rightarrow (1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \left[\beta - \frac{m^2}{1-x^2}\right]y(x) = 0.$$

Observe carefully that, This DE, will match with that associated Legendre DE, (whose solution is known) if $\beta = l(l+1)$, ~~$l \in$~~ $l \in \{0\} \cup \mathbb{Z}^+$

Thus, the solution of this eqn.

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \left[l(l+1) - \frac{m^2}{1-x^2}\right]y(x) = 0$$

is $P_l^m(x) \equiv P_l^m(\cos\theta)$

From Rodrigues formula we have,

$$P_l^m(x) = (1-x^2)^{m/2}\frac{d^m}{dx^m}P_l(x).$$

since $P_l(x)$ is of degree l, we require $0 \le m \le l$. (otherwise all P_l^m with $l < m$ will vanish).
↳ ∵ diff.

Hence the complete angular part solution takes the form. Spherical Harmonics

$$\Theta(\theta)\Phi(\phi) = P_l^m(\cos\theta)\, e^{im\phi} \times N_{lm}.$$
$$= Y_l^m(\theta,\phi).$$

N_{lm} ← Normalization factor.

$m = -l, -l+1, \ldots l-1, l.$

~~$l \in$~~ $l \in \{0\} \cup \mathbb{Z}^+$

Also since, $P_l^{-m}(x) = (-1)^m \frac{(l-m)!}{(l+m)!} P_l^m(x)$

$-m \ge -l$

↳ Physically for opposite Rotations in ϕ

$$N_{lm} = \sqrt{\frac{1}{2\pi}\,\frac{2l+1}{2}\,\frac{(l-m)!}{(l+m)!}}$$

↳ From class

(c) Radial Part:

$$r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} + (k^2r^2 - \beta)R = 0$$

$$\Rightarrow \frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} + \left(k^2 - \frac{l(l+1)}{r^2}\right)R(r) = 0.$$

$$\Rightarrow r^2 \frac{d^2R}{dr^2} + 2r \frac{dR}{dr} + (k^2r^2 - l(l+1))R = 0$$

Substt. $x = kr$.

$$\therefore \frac{dR}{dr} = \frac{dR}{dx}\cdot\frac{dx}{dr} = k\frac{dR}{dx}.$$

$$\frac{d^2R}{dr^2} = k\frac{d}{dx}\left(\frac{dR}{dr}\right) = k^2\frac{d^2R}{dx^2}$$

$$\therefore \boxed{x^2\frac{d^2R}{dx^2} + 2x\frac{dR}{dx} + (x^2 - l(l+1))R(x/k) = 0}$$

↳ This is eqn of spherical bessel function, Now, as l is +ve integer we have,

$$R\left(\frac{x}{k}\right) = A j_l(x) + B n_l(x)$$

$$\Rightarrow \boxed{R(r) = A j_l(kr) + B n_l(kr)}$$

where $k^2 = \frac{2mE}{\hbar^2}$

$A, B \in \mathbb{C}$

↳ Spherical Neumann function.

(d) We know at $r=0$,
$n_l(kr) \rightarrow$ diverges (blows up).
But we want ψ to be finite (from sq. integrability condition)

$\Rightarrow$ Forces $\rightarrow \boxed{B = 0}$

Thus $\boxed{R(r) = A j_l(kr)}$.

(e) Note, $R(r=a) = 0$

$j_l(ka) = 0$.

Let, $ka = b_{nl}$ $\rightarrow$ nth Root l^{th} spherical bessel function.

$$\frac{E_{nl}\,2m\,a^2}{\hbar^2} = b_{nl}^2 \Rightarrow$$

$$\boxed{E_{nl} = \frac{b_{nl}^2\,\hbar^2}{2ma^2}}$$

where $\forall n, j_l(b_{nl}) = 0$ for each $l \in \{0, 1, 2, \ldots\}$.

Thus 'n' corresponds to nth energy level for a particular l-value.

In other words, it corresponds to the nth root of $j_l(x)$.

Note,

n starts from 1, 2, 3, ...

For $l \neq 0$, $j_l(0) = 0 \Rightarrow E_{min} = 0$ not physically acceptable

Emin will be achieved for $(l = 0)$ orbital angular momentum index, as any angular dependence would raise the energy.

$\Rightarrow ka = \alpha$ ← smallest root of $j_0(x)$ which is $\alpha = \pi$.

$\therefore E_{min} = \frac{\pi^2 \hbar^2}{2ma^2} = \frac{h^2}{8ma^2}$

Hence, we are getting.

(F) The general solution is thus,

$$\psi(r, \theta, \phi) = R(r)\,\Theta(\theta)\,\Phi(\phi).$$

$$\psi = \sum_{n=1}^{\infty} \sum_{l=0}^{\infty} A_l\, j_l(k_{nl} r) \sum_{m=-l}^{+l} Y_l^m(\theta, \phi)$$

where $k^2 = \frac{2mE}{\hbar^2}$

Since,

$k_{nl}^2 = \frac{2m E_{nl}}{\hbar^2}$

$\sum_{n=1}^{\infty}$ → arises due to infinite roots of the spherical bessel function

~ physically corresponding to various quantised energy levels.

$\sum_{l=0}^{\infty}$ → Due to sum of all possible values of Angular momentum.

$\sum_{m=-l}^{+l}$ → All possible values of $|L_z|$, corresponding to a certain l.

⑤ Lowest modes

$A_0 \rightarrow$ Normalization constant.

① $\ell = 0, \Rightarrow m = 0$.

$$\psi = A_0 j_0(kr) Y_0^0(\theta, \phi)$$

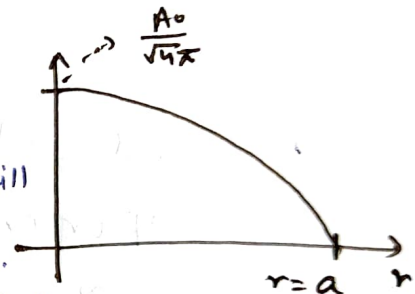
Notice $\psi(r = a, \theta, \phi) = 0$

$\therefore \Rightarrow j_0(ka) = 0$.

$\Rightarrow \dfrac{\sin(ka)}{ka} = 0$ ✓

$\rightarrow k = 0, \dfrac{\pi}{a}, \ldots$

$\therefore \Rightarrow \psi = A_0 \left(\dfrac{\sin\left(\frac{\pi r}{a}\right)}{\frac{\pi r}{a}} \right) \cdot \dfrac{1}{\sqrt{4\pi}}$ ($k > 0$)



(ii) $\ell = 1 \rightarrow m = 0$. $\left\{ \begin{matrix} m=1 \\ m=0 \\ m=-1 \end{matrix} \right\}$ shape will look same in radial part.

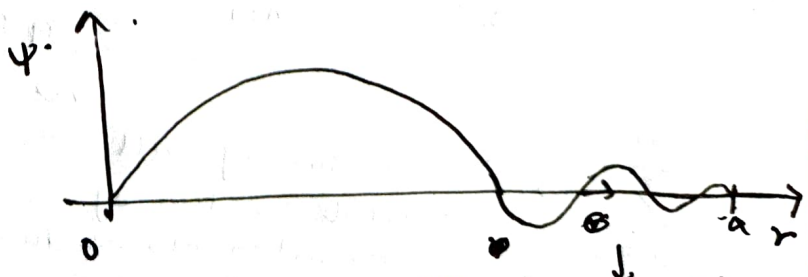
$$\psi = A_1 j_1(kr) Y_1^0(\theta, \phi)$$

$$= A_1 \left(\frac{\sin(kr)}{(kr)^2} - \frac{\cos(kr)}{kr} \right) \sqrt{\frac{3}{4\pi}} \cos\theta$$

$k > 0$. Notice as $r \rightarrow 0$, $j_1(kr) \rightarrow 0$

For a given θ

$\theta \neq \pi/2$



Notice $\psi_1(r = a) = 0$

$\Rightarrow \tan ka = ka$

From computer plotting we find, $ka = 4.49346$

$\psi = A_1 \sqrt{\frac{3}{4\pi}} \cos\theta \sin\left(\frac{4.49346\, r}{a}\right)$

No. of nodes will depend on the value of 'a'.

(6) We have Legendre's DE.

$$\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right] + l(l+1)y = 0$$

l = non-ve, $\in \mathbb{Z}$.

Let $y = \sum_{k=0}^{\infty} a_k x^{k+\lambda}$ $\quad y' = \sum_{k=0}^{\infty} (k+\lambda) a_k x^{k+\lambda-1}$

$$y'' = \sum_{k=0}^{\infty} (k+\lambda)(k+\lambda-1) a_k x^{k+\lambda-2}$$

Plugging in DE,

$$\sum_{k=0}^{\infty} (k+\lambda)(k+\lambda-1) x^{k+\lambda-2} \cdot a_k + \sum_{k=0}^{\infty} \left[l(l+1) - (k+\lambda)(k+\lambda-1) - 2(k+\lambda) \right] a_k x^{k+\lambda} = 0$$

$\therefore$ For $k=0$, we have coeff of lowest power of x.

$$a_0 (\lambda)(\lambda-1) = 0 \qquad (\because a_0 \neq 0)$$

$\Rightarrow \lambda = 1, 0.$

$\therefore$ matching the coefficients of $x^{k+\lambda}$. (We take $\lambda = 0$)

$$a_{k+2} = \frac{-(n-l)(n+l+1)}{(n+2)(n+1)} \cdot a_k$$

coeff. when $x^{\lambda-1}$.

$$(1+\lambda)(\lambda) a_1 = 0.$$

$\lambda = 0$, $a_1 \neq 0$ undetermined.

Hence we set,

$$y(x) = c_0 P_l(x) + c_1 q_l(x)$$

$$P_l(x) = 1 - \frac{l(l+1)}{2!}x^2 + \frac{l(l-2)(l+1)(l+3)}{4!}x^4 + \cdots$$

$$q_l(x) = x - \frac{(l-1)(l+2)}{3!}x^3 + \frac{(l-3)(l-1)(l+2)(l+4)}{5!}x^5 - \cdots$$

Notice, For ℓ a +ve even integer or 0,

P_ℓ terminates.

And for ℓ a +ve odd integer, Q_ℓ terminates after a finite terms.

The one which becomes a polynomial is called $P_\ell(x)$, $\ell = 0, 1, 2, 3, \ldots$

and the other non-terminating series is called $Q_\ell(x)$ (Legendre Functions of 2nd kind).

we previously computed,

$P_0(x) = 1$

$P_1(x) = x$

$P_2(x) = \frac{1}{2}(3x^2 - 1)$

Since one of solutions is known $= y_1(x)$,

The other linearly independent solution,

$$y_2(x) = y_1(x) \int^x \frac{e^{-\int^{x_1} p(x_2)\,dx_2}}{(y_1(x_1))^2}\, dx_1$$

Comparing $p(x_2) = \frac{-2x_2}{1-x^2}$

$$\therefore e^{-\int^{x_1} \frac{2x_2}{1-x^2}\,dx} = e^{-\ln(1-x^2)} = \frac{1}{1-x^2}.$$

For $\ell = 0$, we have,

$$Q_0(z) = P_0(z) \int^z \frac{dx_1}{(1-x_1^2)[P_0(x)]^2}$$

$$= \frac{1}{2}\ln\left(\frac{1+z}{1-z}\right) + A_0.$$

Thus $\therefore \boxed{Q_0(z) = A_0 + B_0 \frac{P_0(z)}{2} \ln\left|\frac{1+z}{1-z}\right|}$

↳ Notice. This blows up at $z = \pm 1$.

A_0, B_0 are constants

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similarly, for $\ell = 1$, we have

$$Q_1(z) = P_1(z) \int^z \frac{dx_1}{(1-x_1^2)x_1^2}$$

$$= P_1(z) \int^z dx_1 \frac{1 - x_1^2 + x_1^2}{(1-x_1^2)(x_1^2)} = z \int^z dx_1 \left(\frac{1}{x_1^2} + \frac{1}{1-x_1^2}\right)$$

Thus,

$$\Rightarrow \boxed{Q_1(z) = A_1 + B_1 z\left(\frac{1}{2}\ln\left|\frac{1+z}{1-z}\right| - \frac{1}{z}\right)}.$$

Similarly (Arfken) other Q_ℓ's can be calculated through Recurrence relations

$$(2n+1)z\,P_n(z) - n\,P_{n-1}(z) = (n+1)P_{n+1}(z)$$