NAME- SUSNATA CHATTOPADHYAY

ROLL - 22MS077

PH 3103 - Assignment (HW) 1

Q.3) $\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + 2y = 0$.

(a) Notice. $P(x) = 2x$, $Q(x) = 2$ are analytic everywhere, so, every finite $x_0 \in \mathbb{R}$ is an ordinary point

To CHECK: At $x_0 = \pm\infty$

Subst. $x = \frac{1}{z}$ $\Big|\therefore \frac{dy}{dx} = \frac{dy}{dz}\left(\frac{dz}{dx}\right) = -z^2\frac{dy}{dz}$

Further,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(-z^2\frac{dy}{dz}\right) = (-z^2)\frac{d}{dz}\left(-z^2\frac{dy}{dz}\right)$$

$$= +2z^3\frac{dy}{dz} + z^4\frac{d^2y}{dz^2}.$$

Thus. transforming $L(x) \longrightarrow L(1/z) = 0$.

we get, $$\frac{d^2y}{dz^2} + \left(\frac{2z-1}{z^2}\right)\frac{dy}{dz} + \frac{2}{z^4}\cdot y = 0.$$

Check at $z \to 0$, $z_0 = 0$

$$\lim_{z\to 0} \tilde{p}(z)(z-z_0) = \lim_{z\to 0}\frac{2z-1}{z} \to \infty \text{ diverges.}$$

So, $z = 0$ OR $x = \pm\infty$ is irregular singular point.

(b) Let $y(x) = \sum_{j=0} c_j x^j$ be a solution around $x_0 = 0$ which is an ordinary point.

$$y' = \sum_{j=1} c_j (j) x^j = \sum_{j=0} c_j (j) x^j$$

$$y'' = \sum_{j=2} j(j-1) c_j x^{j-2} = \sum_{j=0} c_j (j)(j-1) x^{j-2}$$

(Think)

∴ Put.

$$\sum_{j=0} c_j (j)(j-1) x^{j-2} + 2 \sum_{j=0} c_j (j) x^j + 2 \sum_j c_j x^j = 0$$

$$\Rightarrow \sum_{k=0}^{\infty} \left[k(k-1) c_k x^{k-2} + 2 c_k (k) x^k + 2 c_k x^k \right] = 0$$

Equating constant terms i.e powers of x^0.

$$(0 + 2c_0 + 2(2-1) c_2) x^0 = 0$$

$$2c_2 + 2c_0 = 0 \Rightarrow \boxed{c_2 = -c_0}$$

Power of x^1,

$$(3(3-1) c_3 + 2 c_1 (1) + 2 c_1) x^1 = 0$$

$$6 c_3 + 4 c_1 = 0 \rightarrow c_3 = -\frac{4}{6} c_1$$

$$= -\frac{2}{3} c_1$$

So, we have expressions

$$= (2c_2 + 2c_0) + \sum_{k=1}^{\infty} \left[(k+2)(k+1) c_{k+2} + 2 c_k (k) + 2 c_k \right] x^k = 0$$

Thus.

$$(k+2)(k+1) c_{k+2} + 2 c_k (k+1) = 0 \quad \forall k > 1$$

$$\Rightarrow \boxed{c_{k+2} = \frac{-2}{k+2} c_k \quad \forall k.}$$

$$c_4 = \frac{-2}{4} \cdot c_2 = \frac{1}{2} c_0$$

$$c_5 = \frac{-2}{5} c_3 = -\frac{2}{5} \cdot -\frac{2}{3} c_1 = +\frac{4}{15} c_1$$

$$\boxed{\begin{aligned} y(x) = {} & c_0 \left(1 - x^2 + \frac{1}{2} x^4 + \cdots\right) \\ & + c_1 \left(x - \frac{2}{3} x^3 + \frac{4}{15} x^5 - \cdots\right) \end{aligned}}$$

Notice.

$$\left| \frac{c_{k+2}}{c_k} \right| = \frac{2}{k+2} \rightarrow 0 \text{ as } k \rightarrow \infty,$$ Thus, Radius of convergence $= \infty$

Domain of validity of the solution is $\mathbb{R} - \{\infty, -\infty\}$.
$= (-\infty, \infty)$.

(4) $\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + y = 0$.

Notice, $x = 0$ is a Regular singular point.

So, by Frobenius Method. we have,

Let $y(x) = \sum_j a_j x^{j+r}$ $\quad r \in \mathbb{R}$.

Subst.

$$\sum_{j=0} a_j (j+r)(j+r-1) x^{j+r-2} + \sum_{j=0} a_j (j+r) x^{j+r-2} + \sum_{j=0} a_j x^{j+r}$$

$$\Rightarrow \sum_{j=0}^{\infty} \left[\left(a_j (j+r)(j+r-1) + a_j (j+r) \right) x^{j+r-2} + a_j x^{j+r} \right] = 0$$

Solve for coeff of lowest power of x i.e, for $j = 0$.

$$\left(a_0 (r)(r-1) + a_0 (r) \right) x^{r-2} + a_0 x^r = 0.$$

$\because a_0 \neq 0$. we have, $r^2 = 0$

$$\Rightarrow \boxed{r = 0}$$

For $j = 1$, coeff of x^{r-1}

$$a_1 \left[(1+r)(1+r-1) + a(1+r) \right] x^{r-1}$$

put $r = 0$

$$a_1 (1 + 0 + 1) = 0$$

$$\Rightarrow \boxed{a_1 = 0}$$

combining for $j = 2$.

$$= \sum_{j=2}^{\infty} \left[a_{j+2} (j+2)(j+1) + a_{j+2} (j+2) + a_j \right] x^j$$

$$\Rightarrow \sum_{j=2}^{\infty} \left[a_j (j)(j-1) + a_j (j) + a_{j-2} \right] x^{j-2} = 0$$

$$\therefore a_k (k)(k-1) + a_k k + a_{k-2} = 0 \qquad \forall k \geq 2$$

$$\Rightarrow \boxed{a_k = -\frac{1}{k^2} a_{k-2}} \quad \forall k \geq 2$$

Since $a_1 = 0$ all odd terms vanish.

$$a_2 = -\frac{1}{2^2} a_0, \quad a_4 = -\frac{1}{4^2} \cdot a_2 = \frac{1}{4^4} a_0$$

$$a_6 = \frac{-1}{6^2} a_4 = \frac{-1}{6^2 4^2 \cdot 2^2} a_0.$$

$$a_{2\times 3} = \frac{(-1)^3}{2^6 (3!_0)^2} a_0 \Rightarrow a_{2k} = \frac{(-1)^k}{2^{2k} (k!_0)^2} a_0$$

$$\Rightarrow y_1(x) = a_0 \sum_{j=0}^{n} \frac{(-1)^j}{(j!_0)^2} \left(\frac{x^2}{4}\right)^j$$

↳ This nothing but zeroth order Bessel functions, where we take $a_0 = 1$

i.e $y_1(x) = J_0(x)$.

~~$y_1(x)$~~

To Find the $y_2(x)$

(A) WRONSKIAN METHOD.

$$y_2 = y_1 \int_0^x \frac{dx_2 \exp\left[-\int_0^{x_2} P(x_1)\,dx_1\right]}{y_1^2(x_2)}$$

Here we have $P(x_1) = \frac{1}{x_1}$

$$\therefore \exp\left[-\int^{x_2} \frac{dx_1}{x_1}\right] = \frac{1}{x_2}$$

$$\therefore y_2 = y_1 \int^x \frac{dx_2}{x_2} \cdot (y_1)^{-2}$$

From Arfken we have,

$$(y_1)^{-2} = 1 + \frac{x_2^2}{2} + \frac{5}{32} x^4 + \cdots$$

$$\therefore y_2 = J_0(x) \int^x \frac{dx_2}{x_2}\left[1 + \frac{x_2^2}{2} + \frac{5}{32} x_2^4 + \cdots\right]$$

$$= J_0(x) \int^x \left[\frac{1}{x_2} + \frac{x_2}{2} + \frac{5}{32} x_2^3 + \cdots\right] dx_2$$

Integrating,

$$y_2 = J_0(x) \ln x + J_0\left(\frac{x^2}{4} + \frac{5}{128} x^4 + \cdots\right)$$

& $J_0(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64} + \mathcal{O}(x^6)$.

$$\therefore y_2(x) = J_0(x) \ln x + \left(1 - \frac{x^2}{4} + \frac{x^4}{64} + \mathcal{O}(x^6)\right)\left(\frac{x^2}{4} + \cdots\right)$$

$$= J_0(x) \ln x + \left(\frac{x^2}{4} - \frac{3}{128} x^4 + \cdots\right)$$

□

B) Fuch's Theorem

Given, $y_2(x) = \ln(x)\, y_1(x) + \sum_{j=0} c_j x^j$

We check if it's indeed a solution.

$$y_2' = \frac{y_1}{x} + \ln x\, y_1' + \sum_{j=1} c_j (j) x^{j-1}$$

$$y_2'' = -\frac{y_1}{x^2} + \frac{2y_1'}{x} + y_1'' (\ln x) + \sum_{j=2} c_j (j)(j-1) x^{j-2}$$

Notice that,

$$y_1'' + \frac{1}{x} y_1' + y_1 = 0.$$

$\therefore$ Putting in eqn. we have.

$$y_2'' + \frac{1}{x} y_2' + y_2 = 0$$

$$\Rightarrow -\frac{y_1}{x^2} + \frac{2y_1'}{x} + \ln x\,(y_1'') + y_1 \frac{(x)}{x^2} + y_1'(x)\,(\ln x)\, b + \ln x\,(y_1) + \sum_{j=2} c_j (j)(j-1) x^{j-2} + \sum_{j=1} c_j (j) x^{j-2} + \sum_{j=1} c_j x^j = 0.$$

Shifting

$$\sum_{k=0}^{\infty} \left[c_{k+2} (k+1)(k+2) x^k + c_{k+1} (k+1) x^{k-1} + c_{k+1} x^{k+1} \right]$$

$$= -\frac{2}{x} y_1' = \frac{1}{2} \sum_{l=0}^{\infty} \frac{(2l+2)(-1)^l x^{2l}}{2^{2l} ((l+1)!)^2}$$

Equating Coeff both sides

constant term x^0.

$$2c_2 + 2c_2 = \frac{1}{2} \left(\frac{2}{1} \right) \rightarrow \boxed{c_2 = 1/4}$$

For $\frac{1}{x}$, $c_1 = 0$

For all odd powers of x, on LHS, notice.

$$(j+2)(j+1) c_{j+2} + c_{j+2} (j+2) + c_j = 0$$

$$\Rightarrow c_{j+2} = -\frac{c_j}{(j+2)^2}$$

$\therefore$ since $c_1 = 0$, $c_3 = c_5 = \dots = 0$

All odd coeff vanish.

For coeff of x^2.

$$12 c_4 x^2 + 4 c_4 x^2 + c_2 x^2 = \frac{1}{2}\left(\frac{4(-1)}{4 \times 4}\right) x^2 .$$

$$16 c_4 + c_2 = -\frac{1}{8} . \quad \therefore c_2 = 1/4$$

$$16 c_4 = -\frac{1}{8} - \frac{1}{4} = -\frac{3}{8}$$

$$\boxed{c_4 = -3/128}$$

Similarly it can be checked for other powers.

$$\therefore \quad y_2(x) = y_1(x) \ln x + \left(\frac{x^2}{4} - \frac{3}{128} x^4 + \cdots\right)$$

So, we set the explicitly c_j's are same, as that from Wronskian Method.

Q2 $\dfrac{d^2y}{dx^2} + \dfrac{1}{x^2}\dfrac{dy}{dx} - \dfrac{b^2}{x^2} y = 0.$

Ans We see $x = 0$ is a irregular singular point. However, if we still try to find a series solution about $x = 0$, using Frobenius method, let us explore what difficulties do we face :)

Let, $y(x) = \sum_{j=0}^{\infty} a_j x^{j+r}$

$$y' = \sum_{j=0} a_j (j+r) x^{j+r-1}, \quad y'' = \sum_{j=0} a_j (j+r)(j+r-1) x^{j+r-2}$$

$$\therefore \quad x^2 \frac{d^2y}{dx^2} + \frac{dy}{dx} - b^2 y = 0 \quad \text{---- (*)} \quad \text{P.T.O}$$

$$\Rightarrow \sum_{k=0}^{\infty} \left[(k+2)(k+1) a_{k+2} x^{k+2} + (k+1) a_{k+1} x^k - b^2 a_k x^k \right] = 0$$

For $k = 0$.

$$a_1 - b^2 a_0 = 0 \quad \rightarrow a_1 = b^2 a_0$$

For $k = 1$

$$2 a_2 - b^2 a_1 = 0 \quad \rightarrow a_2 = \frac{b^4}{2} a_0$$

Summing up coeff. for x^{k+2}.

$$(k+2)(k+1)\,a_{k+2} + (k+3)\,a_{k+3} - b^2 a_{k+2} = 0 \qquad \forall\ k > 1$$

Recurrence Relation.

$$a_{k+3} = \left[\frac{b^2 - (k+2)(k+1)}{k+3}\right] a_{k+2}.$$

checking convergence, (Ratio Test)

$$\left|\frac{a_{k+3}}{a_{k+2}}\right| = \left|\frac{b^2 - (k+2)(k+1)}{k+3}\right| = \left|\frac{\frac{b^2}{k^2} - \left(1+\frac{2}{k}\right)\left(1+\frac{1}{k}\right)}{\frac{1}{k} + \frac{3}{k^2}}\right|$$

$\longrightarrow \infty$ as $k \to \infty$

Hence, the above series solution does not converge.

From (*) Notice,

~~the coeff the~~ for $j=0$. see the coeff of x^{r-3}.

~~since~~. $a_0(r) = 0$

$\because a_0 \neq 0 \Rightarrow \boxed{r = 0}$

And so, we proceed to the next line.

So, radius of convergence of this solution is zero, and the series doesnot converge.

Hence we don't get a convergent ~~soln~~ series solution around $x_0 = 0$.

Q1) $\frac{d^2y}{dx^2} + w^2 y = 0$.

(a) We take $y(x) = \sum_{j=0}^{\infty} c_j x^j$

Subst. it the above eqn, and we set.

$$\sum_{k=0}^{\infty} \left(c_{k+2}\,(k+2)(k+1) + \omega^2 c_k \right) x^k = 0.$$

Since x^k's are linearly independent

$$\therefore \quad c_{k+2} = -\frac{\omega^2}{(k+1)(k+2)}\, c_k \quad \forall k$$

$k=0$, $\quad c_2 = -\frac{\omega^2}{2!}\, c_0$

$k=1$, $\quad c_3 = -\frac{\omega^2}{3!}\, c_1$

$k=2$

$$c_4 = -\frac{\omega^2}{3\times 4} c_2 = \frac{\omega^4}{4!}\, c_0$$

So,

$$y(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots$$

$$= c_0 \left[1 - \frac{(x\omega)^2}{2!} + \frac{(x\omega)^4}{4!} - \cdots \right]$$

$$+ \frac{c_1}{\omega} \left[\omega x - \frac{\omega^3}{3!} x^3 + \cdots \right]$$

$$\boxed{y(x) = c_0 \cos(\omega x) + \frac{c_1}{\omega} \sin(\omega x).}$$

$$c_0, c_1 \in \mathbb{R}$$

(6) Now we consider,

$$y(x) = \sum_{j=0}^{\infty} c_j (x-1)^j$$

Denote $x' = x - 1$

∴ similarly we have

$$\sum_{k=0}^{\infty} \left[c_{k+2}\,(k+2)(k+1) + \omega^2 c_k \right] (x')^k = 0.$$

we will get the same recurrence relation as before,

And so, $y(x) = c_0 \cos(\omega x') + c_1 \sin(\omega x')$

$$\boxed{y(x) = c_0 \cos(\omega x - \omega) + c_1 \sin(\omega x - \omega)}$$

$$c_0, c_1 \in \mathbb{R}$$

(b) We find the solutions in part (b) are one shifted from that in part (a) by a phase (constant) $= -\omega$.

(c) Let us take $y(x) = x^r \sum_{j=0} c_j x^j$

Put,

$$\sum_{j=0}^{\infty} \left[c_j (j+r)(j+r-1) x^{j+r-2} + \omega^2 c_j (x)^{j+r} \right] = 0.$$

~~set the lowest power~~ coeff of.

set $j = 0$, see lowest power i.e x^{r-2}.

$$c_0 \, r(r-1) = 0$$

$\because c_0 \neq 0$ $\Rightarrow$ $\boxed{r = 0, +1}$

Case 1:

For $r = 0$

$j = 1$ gives.

$$\therefore \quad c_{j+2} = -\frac{\omega^2}{(j+2)(j+1)} c_j \quad \forall j$$

$$\text{i.e } c_2 = -\frac{\omega^2}{2 \cdot 1} c_0 = -\frac{\omega^2}{2!} c_0$$

$$c_3 = -\frac{\omega^2}{3 \cdot 2} c_1 = -\frac{\omega^2}{3!} c_1$$

$$c_4 = -\frac{\omega^2}{4 \cdot 3} \cdot c_2 = +\frac{\omega^4}{4 \cdot 3 \cdot 2 \cdot 1} = \frac{\omega^4}{4!}$$

See the coeff of x^{j+r}.

$$\left[c_{j+2} (j+r+2)(j+r+1) + \omega^2 c_j \right] = 0$$

Put $r = 0$

$$\left[c_{j+2} (j+2)(j+1) + \omega^2 c_j \right] = 0 \quad \forall j$$

(e) c_1 may not be zero,

Thus, we have the solution.

$$y(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

$$= c_0 \left(1 - \frac{\omega^2 x^2}{2!} + \frac{(x\omega)^4 x^4}{4!} + \cdots \right)$$

$$+\frac{c_1}{w}\left(wx - \frac{(xw)^3}{\lfloor 3} + \frac{(xw)^5}{\lfloor 5} + \cdots\right).$$

$$\Rightarrow \boxed{y(x) = c_0 \cos(wx) + c_1 \sin(wx).}$$

For some $c_0, c_1 \in \mathbb{R}$.

Case 2:

for $r = 1$

$$\sum_{j=0}^{\infty} \left[c_j (j+1) j\, x^{j-1} + w^2 c_j x^{j+1}\right] = 0.$$

For $j = 0$, coeff of x^0 (constant):

$$c_1 (2) + \cancel{w^2 c_0} = 0 \Rightarrow \underline{c_1 = 0}$$

Now, collecting coeff of x^{j+1}

$$c_{j+2}\,(j+3)(j+2) + w^2 c_j = 0$$

$$\Rightarrow \quad c_{j+2} = \frac{-w^2}{(j+3)(j+2)} c_j$$

since $c_1 = 0$ we have all odd coeff $(c_3, c_5, \cdots)$ vanish!!

Also since $c_0 \neq 0$:

$$\therefore \quad c_2 = \frac{-w^2}{2 \cdot 3} c_0 = -\frac{w^2}{\lfloor 3} c_0.$$

$$c_4 = -\frac{w^2}{4 \cdot 5} c_2 = +\frac{w^4}{4 \cdot 5 \cdot \lfloor 3} c_0 = +\frac{w^4}{\lfloor 5} c_0.$$

$$\therefore y = x^r\left(c_0 + c_2 x^2 + c_4 x^4 + \cdots\right)$$

$$= \frac{c_0}{w}\left(xw - \frac{(xw)^3}{\lfloor 3} + \frac{(xw)^5}{\lfloor 5} + \cdots\right)$$

$$\boxed{y(x) = \frac{c_0}{w} \sin(xw) = c_0 \sin(wx)}$$

As $c_0 \in \mathbb{R}$ is arbitrary.

Hence we still get the same soln's as in (1) & (6).