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Subj - PH 3102 | Assignment - 06

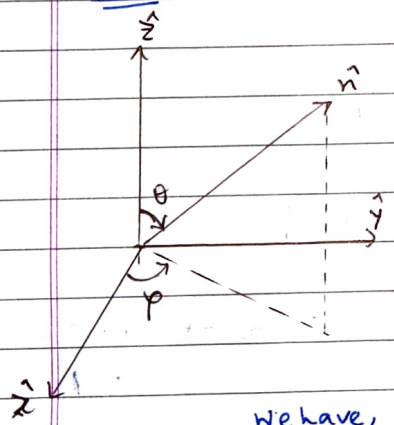
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Q2)

To find eigen states and eigenvalues of spin operator $\vec{S}$ of an e⁻ in a direction given by a unit vector $\vec{n}$: (XZ plane).

Also, Find $P_r\left(\langle S_z \rangle = \hbar/2\right) = ?$

Ans Part-1 $\vec{S}$ in a general $\hat{n}$ direction. $\equiv S_n$



$$\hat{n} = \begin{bmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{bmatrix}$$

$$\vec{S} = \frac{\hbar}{2}\vec{\sigma}$$

We have, $S_n = \vec{S} \cdot \hat{n}$

$$\vec{S} = \frac{\hbar}{2}\left(\sigma_1 \hat{x} + \sigma_2 \hat{y} + \sigma_3 \hat{z}\right)$$

$$= \frac{\hbar}{2}\left[\hat{x}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \hat{y}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + \hat{z}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right)$$

$\therefore$

$$S_n = \frac{\hbar}{2}\begin{bmatrix} \cos\theta & \sin\theta\cos\varphi - i\sin\theta\sin\varphi \\ \sin\theta\cos\varphi + i\sin\theta\sin\varphi & -\cos\theta \end{bmatrix}$$

$$= \frac{\hbar}{2}\begin{bmatrix} \cos\theta & \sin\theta\, e^{-i\varphi} \\ \sin\theta\, e^{i\varphi} & -\cos\theta \end{bmatrix} \equiv S_n(\theta, \varphi)$$

If direction is in XZ plane $\boxed{\varphi = 0}$.

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Part-2 : Eigenvalues of $S_n(\theta,\varphi)$:

Let it's eigenvalue be, $\lambda\frac{\hbar}{2}$.

Then it must be a solution, to the characteristic polynomial of $S_n(\theta,\varphi)$

i.e $\det\left(S_n(\theta,\varphi) - x\,\mathbb{I}\right) = 0$

So

$$\det\left(S_n(\theta,\varphi) - \lambda\frac{\hbar}{2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right) = 0$$

$$\Rightarrow \begin{vmatrix} \cos\theta - \lambda & \sin\theta\, e^{-i\varphi} \\ \sin\theta\, e^{i\varphi} & -\cos\theta - \lambda \end{vmatrix} = 0$$

$$-(\cos^2\theta - \lambda^2) - \sin^2\theta = 0 \Rightarrow -\lambda^2 + 1 = 0$$

$$\lambda = \pm 1$$

$\Rightarrow$ $\Rightarrow$

Eigenvalues of $S_n = +\frac{\hbar}{2}, -\frac{\hbar}{2}$ ∴

which was expected since, we can choose any arbitrary axes, to be S_z, and the eigenvalues should be independent of our choice of coordinates.

Part-3 : Eigenvectors

Let it's eigenvectors be $|v_1\rangle, |v_2\rangle$ → $+\hbar/2$, → $-\hbar/2$

$$\left(S_n(\theta,\varphi) - \frac{\hbar}{2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)|v_1\rangle = \begin{pmatrix}0\\0\end{pmatrix}$$

$$\therefore \begin{pmatrix} \cos\theta - 1 & \sin\theta\, e^{-i\varphi} \\ \sin\theta\, e^{i\varphi} & -\cos\theta - 1 \end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

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$$a(\cos\theta - 1) + b(\sin\theta e^{-i\varphi}) = 0$$
$$a\sin\theta e^{i\varphi} - b(\cos\theta + 1) = 0 \quad \text{---} (*)_1$$

And $\langle v_1 | v_1 \rangle = 1 \Rightarrow |a|^2 + |b|^2 = 1 \quad \text{---} (*)_2$

From $(*)_1$

$$a = \left(\frac{1+\cos\theta}{\sin\theta}\right) e^{-i\varphi} b$$

Put, in $(*)_2$

$$\left[1 + \left(\frac{1+\cos\theta}{\sin\theta}\right)^2\right] |b|^2 = 1$$

$$\Rightarrow |b|^2 = \frac{\sin^2\theta}{2(1+\cos\theta)} \Rightarrow |b| = \sin\frac{\theta}{2} = b$$

(taking +ve value only).

$\therefore$

$$a = \frac{2\cos^2\theta/2}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} \cdot \sin\frac{\theta}{2} e^{-i\varphi}$$

$$= \cos\theta/2\, e^{-i\varphi}$$

$$\therefore |v_1\rangle = \begin{pmatrix} \cos\theta/2\, e^{-i\varphi} \\ \sin\theta/2 \end{pmatrix}$$ Ans

And, other one. $|v_2\rangle = \begin{pmatrix} c \\ d \end{pmatrix}$

$$\begin{pmatrix} \cos\theta + 1 & \sin\theta e^{-i\varphi} \\ \sin\theta e^{i\varphi} & +\cos\theta + 1 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$c(\cos\theta + 1) + d\sin\theta e^{-i\varphi} = 0$$
$$c(\sin\theta e^{i\varphi}) + d(1 - \cos\theta) = 0 \quad \text{---} (*)_3$$

And

$$\langle v_2 | v_2 \rangle = 1 \Rightarrow |c|^2 + |d|^2 = 1 \quad \text{---} (*)_4$$

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similarly solving (*3), (*4), we find

$$|v_2\rangle = \begin{pmatrix} -\sin\theta/2\, e^{-i\varphi} \\ \cos\theta/2 \end{pmatrix}$$

For x-z plane, $\varphi = 0$

∴ Eigen vectors of $S_n(\theta, 0)$ are -

$$|v_+\rangle = \begin{pmatrix} \cos\theta/2 \\ \sin\theta/2 \end{pmatrix} \quad |v_-\rangle = \begin{pmatrix} -\sin\theta/2 \\ \cos\theta/2 \end{pmatrix}$$

$|v_+\rangle \downarrow +\hbar/2$ $\quad$ $|v_-\rangle \downarrow -\hbar/2$

Ans

since they also form an eigenbasis of S_n

Part-4 $\quad Pr(S_z = +\hbar/2) = ?$

Any general spin state $|\alpha\rangle$ can be written as

$$|\alpha\rangle = c_+ |v_+\rangle + c_- |v_-\rangle.$$

The eigenvalues of S_z are $|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |+\frac{1}{2}\rangle$

And, $|\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |-\frac{1}{2}\rangle$

Since, c_+, c_- can be arbitrary, we compute the respective probabilities for $|v_+\rangle, |v_-\rangle$ only.

Note, $|v_+\rangle = \cos\frac{\theta}{2}|\uparrow\rangle + \sin\frac{\theta}{2}|\downarrow\rangle$

$$\therefore P(S_z = +\hbar/2) = \frac{|\langle\uparrow|v_+\rangle|^2}{\langle v_+|v_+\rangle} = \cos^2\frac{\theta}{2}$$ Ans

~~Hint~~ The probability of measuring $\langle \hat{S}_z \rangle = \hbar/2$ for $|v_+\rangle$ state is $\cos^2\theta/2$.

And,

$$|v_-\rangle = -\sin\frac{\theta}{2}|\uparrow\rangle + \cos\frac{\theta}{2}|\downarrow\rangle$$

$$\therefore P(S_z = -\hbar/2) = \frac{|\langle\uparrow|v_-\rangle|^2}{\langle v_-|v_-\rangle} = \boxed{\sin^2\theta/2}$$ Ans

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Q3) Given: $j_1 = 3/2$, $j_2 = 1/2$

(a) To compute all eigenvectors of $\vec{J} = \vec{J}_1 + \vec{J}_2$

A. A system of two angular momenta, of respective magnitudes $j_1 = 3/2$, $j_2 = 1/2$ is described by the basis $|j_1 = 3/2, m_1\rangle \otimes |j_2 = 1/2, m_2\rangle$ — ①

The possible values of (m_1, m_2) are

$$\begin{cases} j_1 = 3/2 \rightarrow m_1 = 3/2, 1/2, -1/2, -3/2 \\ j_2 = 1/2 \rightarrow m_2 = 1/2, -1/2 \end{cases}$$

Now, we know that in particular, the total angular momenta, can take the values

$$J = |j_1 - j_2|, |j_1 - j_2| + 1, \ldots, j_1 + j_2$$
$$= 1, 2$$

And the z-component of $\vec{J}$, or M is

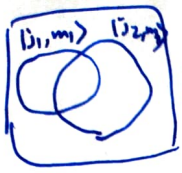
$$\begin{cases} J = 1 \rightarrow M = 1, 0, -1 \\ J = 2 \rightarrow M = 2, 1, 0, -1, -2 \end{cases}$$

So, for the combined states there are a total of $(2j_1+1)(2j_2+1) = (3+1)(1+1) = 8$ possible eigenstates in a basis of combined states (coupled basis).

From ①, we have

$$|m_1, m_2\rangle \equiv |j_1, m_1; j_2, m_2\rangle = |j_1, m_1\rangle \otimes |j_2, m_2\rangle$$

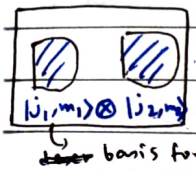
The subspace spanned by this product basis states $(|m_1, m_2\rangle)$, is the subspace in which the desired eigenstates of the coupled system $(|J, M\rangle)$ exists. $|J, M\rangle$'s are called coupled basis states.



As $[J_1, J_2] = 0$

vectors





basis for both spaces. (disjoint)

This where $|J,M\rangle$ coupled basis for the system lies.

$\{|J,M\rangle\} \xrightarrow{C.G} \{|m_1,m_2\rangle\}$

coupled basis — product basis

"Change of basis"

The coordinates of $|J,M\rangle$ w.r.t $|m_1,m_2\rangle$ basis states are what is called Clebsch-Gordan (C.G) coefficients.

From the completeness of basis states we can write

$$\sum_{J,M} |J,M\rangle\langle J,M| = \mathbb{I}$$

&

Since, j_1, j_2 are fixed

$$\sum_{m_1,m_2} |m_1,m_2\rangle\langle m_1,m_2|$$

$$= \sum_{m_1,m_2} |j_1,m_1;j_2,m_2\rangle\langle j_1,m_1;j_2,m_2| = \mathbb{I}$$

i.e The eigenstates can be written as,

$$|J,M\rangle = \mathbb{I}\,|J,M\rangle$$

$$= \sum_{m_1,m_2 \text{ or } m_1+m_2=M} \underbrace{\langle j_1,m_1;j_2,m_2|J,M\rangle}_{C.G} |j_1,m_1;j_2,m_2\rangle$$

because we have

$\Rightarrow \langle j_1,m_1;j_2,m_2|J,M\rangle = 0$ if $m_1+m_2 \neq M$

Also, note $\langle m_1,m_2| \equiv \langle j_1,m_1;j_2,m_2|$ ($\because j_1, j_2 \rightarrow$ always fixed)

$$\Rightarrow \langle m_1,m_2|J,M\rangle = (-1)^{j_1+j_2-J} \langle -m_1,-m_2|J,-M\rangle$$

For each $J = \{1,2\}$, we shall start from

$|J, M=J\rangle \longrightarrow |J, M=0\rangle$

- $\hat{J}_\pm |J,M\rangle = \hbar\sqrt{(J\mp M)(J\pm M+1)}\,|J,M\pm1\rangle$
- $\hat{J}_{1-} |m_1,m_2\rangle = \hbar\sqrt{(j_1+m_1)(j_1-m_1+1)}\,|m_1-1,m_2\rangle$
- $\hat{J}_{2-} |m_1,m_2\rangle = \hbar\sqrt{(j_2+m_2)(j_2-m_2+1)}\,|m_1,m_2-1\rangle$

Q.3) (a) $j_1 = 3/2, \quad j_2 = 1/2$

To write the eigenstates $|J, M\rangle$ (coupled basis) in terms of $|j_1, m_1 ; j_2, m_2\rangle \equiv |m_1, m_2\rangle$ (product basis)

(i) We start with $J = j_1 + j_2 = 2$ sector.

And $M \to M_{max} = J = 2$

the corresponding $|j_1, m_1 ; j_2, m_2\rangle$ state can be only when $m_1 + m_2 = 2$

$m_1 = 3/2$ and $m_2 = 1/2$

$\therefore |J=2, M=2\rangle = \left|m_1 = \frac{3}{2}, m_2 = \frac{1}{2}\right\rangle$

↳ Both are normalised eigenstates!

(ii) Now act with $\hat{J}_- = \hat{J}_{1-} + \hat{J}_{2-}$.

$$\hat{J}_- |J=2, M=2\rangle = (J_{1-} + J_{2-}) \left|m_1 = \frac{3}{2}, m_2 = \frac{1}{2}\right\rangle$$

$$= J_{1-} \left|m_1 = \frac{3}{2}, m_2 = \frac{1}{2}\right\rangle + J_{2-} \left|m_1 = \frac{3}{2}, m_2 = \frac{1}{2}\right\rangle$$

$$\Rightarrow \cancel{\sqrt{(2+2)\cdot 1}} \; |J=2, M=1\rangle$$

$$= \cancel{\sqrt{3}} \left|m_1 = \frac{1}{2}, m_2 = \frac{1}{2}\right\rangle + \cancel{1} \left|m_1 = \frac{3}{2}, m_2 = -\frac{1}{2}\right\rangle$$

$$\therefore |J=2, M=1\rangle = \frac{\sqrt{3}}{2} \left|m_1 = \frac{1}{2}, m_2 = \frac{1}{2}\right\rangle + \frac{1}{2} \left|m_1 = \frac{3}{2}, m_2 = -\frac{1}{2}\right\rangle$$

(iii) Again J_-.

$$\hat{J}_- |J=2, M=1\rangle = \frac{\sqrt{3}}{2} J_{1-} \left|m_1 = \frac{1}{2}, m_2 = \frac{1}{2}\right\rangle + \frac{\sqrt{3}}{2} J_{2-} \left|m_1 = \frac{1}{2}, m_2 = 1/2\right\rangle + \frac{1}{2} J_{1-} \left|m_1 = \frac{3}{2}, m_2 = -1/2\right\rangle + \frac{1}{2} J_{2-} \left|m_1 = \frac{3}{2}, m_2 = -1/2\right\rangle$$

$= 0$

$\frac{1}{\sqrt{6}} |J=2, M=0\rangle$

$= \frac{\sqrt{3}}{2} \cdot \sqrt{2} \cdot \frac{1}{\sqrt{6}} \left| m_1 = -\frac{1}{2}, m_2 = \frac{1}{2} \right\rangle$

$+ \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{6}} \left| m_1 = \frac{1}{2}, m_2 = -\frac{1}{2} \right\rangle$

$+ \frac{1}{2} \cdot \frac{1}{\sqrt{6}} \cdot \sqrt{3} \left| m_1 = \frac{1}{2}, m_2 = -\frac{1}{2} \right\rangle$

$\therefore \boxed{|J=2, M=0\rangle = \frac{1}{\sqrt{2}} \left| m_1 = -\frac{1}{2}, m_2 = \frac{1}{2} \right\rangle + \frac{1}{\sqrt{2}} \left| m_1 = \frac{1}{2}, m_2 = -\frac{1}{2} \right\rangle}$

(iv) Let,

$|J=2, M=-1\rangle = C_1 \left| m_1 = -\frac{1}{2}, m_2 = -\frac{1}{2} \right\rangle + C_2 \left| m_1 = -\frac{3}{2}, m_2 = \frac{1}{2} \right\rangle$

By relation we have,

$C_1 = \left\langle m_1 = -\frac{1}{2}, m_2 = -\frac{1}{2} \middle| J=2, M=-1 \right\rangle = \left\langle m_1 = \frac{1}{2}, m_2 = \frac{1}{2} \middle| J=2, M=1 \right\rangle$

$(-1)^{j_1+j_2-J}$

$= \frac{\sqrt{3}}{2}$ $\quad (\because j_1+j_2 = 1 = 2)$

&

$C_2 = \left\langle m_1 = -\frac{3}{2}, m_2 = \frac{1}{2} \middle| J=2, M=-1 \right\rangle = \left\langle m_1 = \frac{3}{2}, m_2 = -\frac{1}{2} \middle| J=2, M=+1 \right\rangle$

$= \frac{1}{2}$

$\therefore \boxed{|J=2, M=-1\rangle = \frac{\sqrt{3}}{2} \left| m_1 = -\frac{1}{2}, m_2 = -\frac{1}{2} \right\rangle + \frac{1}{2} \left| m_1 = -\frac{3}{2}, m_2 = \frac{1}{2} \right\rangle}$

(v) Now,

$|J=2, M=-2\rangle$ is possible only if

$m_1 = -\frac{1}{2}, m_2 = -\frac{3}{2}$

$\therefore |J=2, M=-1\rangle = \left| m_1 = -\frac{1}{2}, m_2 = -\frac{3}{2} \right\rangle$

(vi) Now, we begin $J=1$ sector.

$|J=1, M=1\rangle = C_1 \left| m_1 = \frac{1}{2}, m_2 = \frac{1}{2} \right\rangle + C_2 \left| m_1 = \frac{3}{2}, m_2 = -\frac{1}{2} \right\rangle$

Normalisation condition gives $C_1^2 + C_2^2 = 1$.

Now consider, orthogonality of $|J=2, M=1\rangle$ & $|J=1, M=1\rangle$

$\langle 2,1 | 1,1 \rangle = 0 \Rightarrow \sqrt{3} C_1 = -C_2$

Thm

$C_1 = \frac{1}{2}, \quad C_2 = -\frac{\sqrt{3}}{2}$

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$\therefore$

$$|J=1, M=1\rangle = \frac{1}{\sqrt{2}}\left(|m_1=\tfrac{1}{2}, m_2=\tfrac{1}{2}\rangle - \sqrt{3}\,|m_1=\tfrac{3}{2}, m_2=-\tfrac{1}{2}\rangle\right)$$

(vii) Act $\hat{J}_-$.

$$\hat{J}_-|J=1,M=1\rangle = \frac{1}{\sqrt{2}} J_{1-}\left|1, \tfrac{1}{2}, \tfrac{1}{2}\right\rangle + \frac{J_{2-}}{\sqrt{2}}\left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle$$

$$-\frac{\sqrt{3}}{2} J_{1-}\left|\tfrac{3}{2}, -\tfrac{1}{2}\right\rangle - \frac{J_{2-}}{\sqrt{2}}\left|\tfrac{3}{2}, -\tfrac{1}{2}\right\rangle \frac{\sqrt{3}}{2}.$$

$\Rightarrow$ $\sqrt{2}$ $|J=1, M=0\rangle$

$$= \frac{1}{\sqrt{2}} \hbar\sqrt{2}\left|-\tfrac{1}{2}, \tfrac{1}{2}\right\rangle + \frac{\hbar}{\sqrt{2}}\left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle$$

$$-\frac{\sqrt{3}}{2} \hbar\sqrt{3}\left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle$$

$$\therefore |J=1, M=0\rangle = \frac{1}{\sqrt{2}}\left|-\tfrac{1}{2}, \tfrac{1}{2}\right\rangle + \left(-\frac{1}{2}\right)\frac{-1}{\sqrt{2}}\left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle$$

$$= \frac{1}{\sqrt{2}}\left(\left|-\tfrac{1}{2}, \tfrac{1}{2}\right\rangle + \left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle\right)$$

(viii) Similarly, we let

$$|J=1, M=-1\rangle = c_1\left|-\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle + c_2\left|-\tfrac{3}{2}, \tfrac{1}{2}\right\rangle$$

$$\therefore c_1 = \langle m_1=-\tfrac{1}{2}, m_2=-\tfrac{1}{2}\,|\,J=1, M=-1\rangle$$

$$= \frac{\langle m_1=\frac{1}{2}, m_2=\frac{1}{2}\,|\,J=1, M=1\rangle}{(-1)^{2-J}}$$

$$= -1/\sqrt{2}$$

And

$$c_2 = +1/\sqrt{2} = -\langle m_1=+\tfrac{3}{2}, m_2=-\tfrac{1}{2}\,|\,J=1, M=1\rangle$$

$$\therefore |J=1, M=-1\rangle = -\frac{1}{\sqrt{2}}\left|-\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle + \frac{1}{\sqrt{2}}\left|-\tfrac{3}{2}, \tfrac{1}{2}\right\rangle.$$

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Summarising.

$$j_1 = \frac{3}{2},\ j_2 = \frac{1}{2}$$

	$\lvert J, M\rangle$	$\lvert j_1, m_1\rangle \otimes \lvert j_2, m_2\rangle \equiv \lvert m_1, m_2\rangle$ base.
1)	$\lvert 2, 2\rangle$	$\left\lvert \frac{3}{2}, \frac{1}{2}\right\rangle$
2)	$\lvert 2, 1\rangle$	$\frac{\sqrt{3}}{2}\left\lvert \frac{1}{2}, \frac{1}{2}\right\rangle + \frac{1}{2}\left\lvert \frac{3}{2}, -\frac{1}{2}\right\rangle$
3)	$\lvert 2, 0\rangle$	$\frac{1}{\sqrt{2}}\left\lvert -\frac{1}{2}, \frac{1}{2}\right\rangle + \frac{1}{\sqrt{2}}\left\lvert \frac{1}{2}, -\frac{1}{2}\right\rangle$
4)	$\lvert 2, -1\rangle$	$\frac{\sqrt{3}}{2}\left\lvert -\frac{1}{2}, -\frac{1}{2}\right\rangle + \frac{1}{2}\left\lvert -\frac{3}{2}, \frac{1}{2}\right\rangle$
5)	$\lvert 2, -2\rangle$	$\left\lvert -\frac{3}{2}, -\frac{1}{2}\right\rangle$
6)	$\lvert 1, 1\rangle$	$\frac{1}{2}\left\lvert \frac{1}{2}, \frac{1}{2}\right\rangle - \frac{\sqrt{3}}{2}\left\lvert \frac{3}{2}, -\frac{1}{2}\right\rangle$
7)	$\lvert 1, 0\rangle$	$\frac{1}{\sqrt{2}}\left\lvert -\frac{1}{2}, \frac{1}{2}\right\rangle - \frac{1}{\sqrt{2}}\left\lvert \frac{1}{2}, -\frac{1}{2}\right\rangle$
8)	$\lvert 1, -1\rangle$	$-\frac{1}{\sqrt{2}}\left\lvert -\frac{1}{2}, -\frac{1}{2}\right\rangle + \frac{1}{\sqrt{2}}\left\lvert -\frac{3}{2}, \frac{1}{2}\right\rangle$

(5) $\vec{J_1}\cdot\vec{J_2}$ =?

$$\vec{J_1}\cdot\vec{J_2} = J_{1x}J_{2x} + J_{1y}J_{2y} + J_{1z}J_{2z}$$

and

$$\because\ J_{n\pm} = J_{nx} \pm i J_{ny} \qquad n \in \{1,2,3\}$$

we can deduce.

$$\vec{J_1}\cdot\vec{J_2} = J_{1z}J_{2z} + \frac{1}{2}\left(J_{1-}J_{2+} + J_{1+}J_{2-}\right)$$

$$J_{1z}\lvert m_1, m_2\rangle = m_1\hbar\lvert m_1, m_2\rangle$$
$$J_{1-}\lvert m_1, m_2\rangle = \sqrt{(j_1+m_1)(j_1-m_1+1)}\ \lvert m_1-1, m_2\rangle$$

$$m_1 \in \left\{\tfrac{3}{2}, \tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{3}{2}\right\},\ m_2 \in \left\{\tfrac{1}{2}, -\tfrac{1}{2}\right\}$$

M=

So,

$\vec{J_1}\cdot\vec{J_2}$

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But, $\vec{J} = \vec{J}_1 + \vec{J}_2$

$\therefore J^2 = J_1^2 + J_2^2 + 2\vec{J}_1 \cdot \vec{J}_2$

$\therefore \vec{J}_1 \cdot \vec{J}_2 = \frac{1}{2}\left(J^2 - J_1^2 - J_2^2\right)$

$\hat{J}^2 |J,M\rangle = J(J+1)\hbar^2 |J,M\rangle$

$J_1^2 |m_1,m_2\rangle = j_1(j_1+1)\hbar^2 |m_1,m_2\rangle = \frac{15}{4}\hbar^2 |m_1,m_2\rangle$

$J_2^2 |m_1,m_2\rangle = j_2(j_2+1)\hbar^2 |m_1,m_2\rangle = \frac{3}{4}\hbar^2 |m_1,m_2\rangle$

(i) $J_1 \cdot J_2 |2,2\rangle = \frac{1}{2}\left[6\hbar^2 |2,2\rangle - \left(\frac{15}{4}\hbar^2 + \frac{3}{4}\hbar^2\right)|2,2\rangle\right]$

$= \frac{3}{4}\hbar^2 |2,2\rangle$

(ii) $J_1 \cdot J_2 |2,1\rangle = \frac{1}{2}\left[6\hbar^2 |2,1\rangle - \frac{\sqrt{3}}{2}\times\frac{15}{4}\hbar^2 \left|\frac{1}{2},\frac{1}{2}\right\rangle - \frac{\sqrt{3}}{2}\cdot\frac{3}{4}\hbar^2 \left|\frac{1}{2},\frac{1}{2}\right\rangle - \frac{1}{2}\cdot\frac{15}{4}\hbar^2 \left|\frac{3}{2},\frac{-1}{2}\right\rangle - \frac{3}{4}\hbar^2 \times \frac{1}{2}\left|\frac{3}{2},\frac{-1}{2}\right\rangle\right]$

$= \frac{\hbar^2}{2}\left[6|2,1\rangle - \frac{15\sqrt{3}}{8}\left|\frac{1}{2},\frac{1}{2}\right\rangle - \frac{3\sqrt{3}}{8}\left|\frac{1}{2},\frac{1}{2}\right\rangle - \frac{15}{8}\left|\frac{3}{2},\frac{-1}{2}\right\rangle - \frac{3}{8}\left|\frac{3}{2},\frac{-1}{2}\right\rangle\right]$

$= \frac{\hbar^2}{2}\left[6|2,1\rangle - \frac{9\sqrt{3}}{4}\left|\frac{1}{2},\frac{1}{2}\right\rangle - \frac{9}{4}\left|\frac{3}{2},\frac{-1}{2}\right\rangle\right]$

$= \frac{\hbar^2}{2}\left[6|2,1\rangle - \frac{9}{2}|2,1\rangle\right] = \frac{3}{4}\hbar^2 |2,1\rangle$

$\therefore \vec{J}_1 \cdot \vec{J}_2 |J,M\rangle = \frac{3}{4}\hbar^2 |J,M\rangle \quad \forall M, \; J=2$

↳ depends only on J

Similarly.

$M = j_z$.

$J_1 \cdot J_2 |1,0\rangle = \frac{\hbar^2}{2}\left(2|1,0\rangle - \frac{1}{\sqrt{2}}\frac{15}{4}\left|\frac{-1}{2},\frac{1}{2}\right\rangle - \frac{1}{\sqrt{2}}\frac{3}{4}\left|\frac{-1}{2},\frac{1}{2}\right\rangle + \frac{1}{\sqrt{2}}\frac{15}{4}\left|\frac{1}{2},\frac{-1}{2}\right\rangle + \frac{1}{\sqrt{2}}\frac{3}{4}\left|\frac{1}{2},\frac{-1}{2}\right\rangle\right)$

$= -\frac{5}{4}\hbar^2 |1,0\rangle$

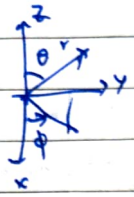
So, Ans

$$\vec{J}_1 \cdot \vec{J}_2 |J,M\rangle = \begin{cases} \frac{3}{4}\hbar^2 |J=2,M\rangle & M \in \{-2,\dots,2\} \\ -\frac{5}{4}\hbar^2 |J=1,M\rangle & M \in \{-1,0,1\} \end{cases}$$

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Q1) $\psi(x,y,z) = (x+y+3z)\, f(r)$.

Since, $V(r)$ is spherically symmetric, we resort to transform the coordinates from cartesian to polar.



$$z = r\cos\theta$$
$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$

$$\therefore \psi(r,\theta,\phi) = r(3\cos\theta + \sin\theta\cos\phi + \sin\theta\sin\phi) f(r)$$
$$= R(r)\,\Theta(\theta,\phi)$$

(a) $$L^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$$

Checking if it's an eigenfunction.

$$L^2\psi = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}\right]$$ → solution is $Y_{l,m}(\theta,\phi)$.

Since, $[L^2, L_z] = 0$, we know ~~solutions~~ eigenfunctions of

L^2 are spherical harmonics – $Y_{l,m}(\theta,\phi)$
↳ (only for central potential $V(r)$) $\sim P_{l,m}(\cos\theta)\, e^{im\phi}$

And we have defined them such that.

$$L^2 Y_{l,m}(\theta,\phi) = \hbar^2 l(l+1)\, Y_{l,m}(\theta,\phi)$$
$$L_z Y_{l,m}(\theta,\phi) = m\hbar\, Y_{l,m}(\theta,\phi)$$

Now, Associated Legendre Polynomials are

$$P_{l,m}(\cos\theta) = (\sin\theta)^m \frac{d^m}{d(\cos\theta)^m}\left(P_l(\cos\theta)\right)$$

$P_l(\cos\theta)$ → Legendre Polynomial so, $\sim (\cos\theta)^l$

$\therefore \frac{d^m}{d(\cos\theta)^m} P_l(\cos\theta) \sim (\cos\theta)^{l-m}$ (After different m times)

$\therefore Y_{l,m}(\theta,\phi) \sim (\sin\theta)^m (\cos\theta)^{l-m} e^{im\phi}$

Looking at the angular part of ψ,

$= 3\cos\theta + \sin\theta(\cos\phi) + \sin\theta\sin\phi$

compare powers of $\sin\theta$ $\cos\theta$

$3\cos\theta$	$\sin\theta(\cos\phi)$	$\sin\theta\sin\phi$
$m=0$	$m=1$	$m=1$
$l-m=1 \Rightarrow l=1$	$l-m=0 \Rightarrow l=1$	$l-m=0 \Rightarrow l=1$

Since, $l=1$ is common in all, we check the spherical Harmonics for which $l=1$

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$$Y_{1,0} = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_{1,\pm1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{\pm i\phi}$$

Denote $K = \sqrt{\frac{3}{8\pi}}$

∴ Angular part $\Theta(\theta,\phi)$ can be written as.

$$\Theta = C_0 Y_{1,0} + C_1 Y_{1,1} + C_2 Y_{1,-1}$$

$$= \sqrt{8}\, C_0 K\sqrt{2}\cos\theta + C_1(-K\sin\theta)(\cos\phi + i\sin\phi) + C_{-1}(K\sin\theta)(\cos\phi - i\sin\phi)$$

comparing,

$$\left.\begin{aligned} C_0 K\sqrt{2} &= 3 \\ -KC_1 + KC_{-1} &= 1 \\ -iKC_1 - iKC_{-1} &= 1 \end{aligned}\right\} \quad \begin{aligned} C_0 &= \frac{3}{K\sqrt{2}} \\ C_1 &= -\frac{1}{K\sqrt{2}} e^{-i\pi/4} \\ C_{-1} &= \frac{1}{K\sqrt{2}} e^{i\pi/4} \end{aligned}$$

Thus, we have.

$$\Psi = R(r)\left[\frac{3}{K\sqrt{2}} Y_{1,0} - \frac{1}{K\sqrt{2}} e^{-i\pi/4} Y_{1,1} + \frac{1}{K\sqrt{2}} e^{i\pi/4} Y_{1,-1}\right]$$

since $\forall\, l = 1$

$$L^2 Y_{l,m}(\theta,\phi) = 2\hbar^2 Y_{l,m}(\theta,\phi)$$

$$\Rightarrow \boxed{L^2\Psi = 2\hbar^2\Psi}$$

And so, the given $\Psi(r,\theta,\phi)$ or $(\Psi(x,y,z))$ is indeed an eigenfunction of L^2, with the 'good quantum number' anociated, is $\boxed{l = 1}$ ✓

(b) $$|C_0|^2 + |C_1|^2 + |C_{-1}|^2 = \|\Theta\|^2$$

$$= \frac{1}{K^2}\left(\frac{9}{2} + \frac{1}{2} + \frac{1}{2}\right) = \frac{11}{2K^2}$$

$$= 4\cdot\frac{8\pi}{3}\left(\frac{11}{2}\right) = \frac{44\pi}{3}.$$

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∴ Probability of the particle to be present in m state is

$$P(m=0) \rightarrow \frac{|C_0|^2}{\|\Psi\|^2} = \frac{9/2k^2}{11/2k^2} = \frac{9}{11}$$

$$P(m=-1) \rightarrow \frac{|C_{-1}|^2}{\|\Psi\|^2} = \frac{1/2k^2}{11/2k^2} = \frac{1}{11}$$

$$P(m=1) \rightarrow \frac{|C_1|^2}{\|\Psi\|^2} = \frac{1/2k^2}{11/2k^2} = \frac{1}{11}$$

Ans

(c) Given $\psi(x,y,z)$ is an energy eigenfunction with eigenvalue E.
$V(r) = ?$

Ans We begin by writing the Hamiltonian of system

$$\hat{H} = -\frac{\hbar^2}{2m}\nabla^2 + V(r)$$

in sph. coord.

$$\nabla^2 = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r) + \frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$$

$$= \frac{1}{r}\frac{\partial^2}{\partial r^2}(r) - \frac{\hat{L}^2}{\hbar^2 r^2}$$

Thus the differential eqn for the stationary wavefn. $\Psi(r,\theta,\phi)$ is

$$\hat{H}\Psi = -\frac{\hbar^2}{2m}\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Psi) + \frac{\hat{L}^2\Psi}{2mr^2} + V(r)\Psi = E\Psi$$

$$= -\frac{\hbar^2}{2m}\cdot\Theta(\theta,\phi)\cdot\frac{1}{r}\left[2f(r) + 4r\frac{\partial f}{\partial r} + r^2\frac{\partial f}{\partial r^2}\right] + \frac{\hbar^2}{2mr^2}f(r)\Theta(\theta,\phi) + rf(r)\Theta V(r) = Erf(r)\Theta$$

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$$= -\frac{\hbar^2}{2m}\left(\frac{2f}{r} + \frac{2}{r}\frac{\partial f}{\partial r} + \frac{r}{2}\frac{\partial^2 f}{\partial r^2}\right) + \frac{\hbar^2}{m}\frac{f}{r}.$$

$$+ r f V(r) = E r f(r)$$

$$\therefore \quad -\frac{\hbar^2}{m r f(r)}\left[2\frac{\partial f}{\partial r} + \frac{r}{2}\frac{\partial^2 f}{\partial r^2}\right] - E = V(r).$$

↓

solution to this differential equation shall yield us the potential $V(r)$.