PH 3102

Name - SUSNATA CHATTOPADHYAY
Roll - 22MS077 , Assignment - 4

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Q3)

(a) $i\hbar \frac{\partial \Psi(x,t)}{\partial t} = \hat{H}\, \Psi(x,t)$

To show $\Psi(x,t)$ remains ~~eigenstate~~ coherent state.

Ans We obtained,

$$\Psi(x,0) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{|\alpha|^2}{2}} \cdot \exp\left(\frac{m\omega}{\hbar}\left(\alpha x - \frac{x^2}{2}\right)\right)$$

where $r = \sqrt{\frac{m\omega}{2\hbar}}$.

From given eqn we know.

$$\Psi(x,t) = \hat{U}_t\, \Psi(x,0) \;(U_t)^* \Rightarrow |\Psi;t\rangle = \hat{U}_t |\Psi;0\rangle.$$
$$= \exp\left(\frac{-i\hat{H}t}{\hbar}\right) \Psi(x,0).$$

$$\Psi(x,0) = \sum_{n=0}^{\infty} C_n \psi_n(x)$$

$$\Rightarrow |\Psi;0\rangle = \sum_n \frac{\alpha^n}{\sqrt{n!}} C_0 |n\rangle$$

$$\therefore |\Psi;t\rangle = e^{\frac{-i\hat{H}t}{\hbar}} |\Psi;0\rangle = \sum_n \frac{\alpha^n}{\sqrt{n!}} C_0\, e^{\frac{-iE_n t}{\hbar}} |n\rangle$$

where $E_n = \left(n + \frac{1}{2}\right)\hbar\omega$ of SHO.

we had, $\hat{a}\,|\Psi;0\rangle = \alpha\,|\Psi;0\rangle$

↳ definition of ~~the~~ coherent state.

To show,

$$\hat{a}\,|\Psi;t\rangle \overset{?}{=} \alpha(t)\,|\Psi;t\rangle$$

And, $\alpha(t) \overset{?}{=} \alpha_0$

LHS

$$\sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \, c_n e^{-\frac{iE_n t}{\hbar}} (\hat{a}|n\rangle)$$

$$\hat{a}|n\rangle \to \sqrt{n}\,|n-1\rangle.$$

$$= \sum_{n=1}^{\infty} \frac{\alpha^n}{\sqrt{(n-1)!}} \, c_n e^{-\frac{iE_n t}{\hbar}} |n-1\rangle.$$

Subs. $n-1 = k$

$$= \sum_{k=0}^{\infty} \frac{\alpha^{k+1}}{\sqrt{k!}} \, c_{k+1} e^{-\frac{iE_{k+1} t}{\hbar}} |k\rangle$$

Notice that

$$E_{k+1} = E_k + \hbar\omega.$$

Thus

$$= \left(\alpha e^{-i\omega t}\right) \sum_{k=0}^{\infty} \frac{\alpha^k}{\sqrt{k!}} \, c_k e^{-\frac{iE_k t}{\hbar}} |k\rangle$$

$$= \alpha(t)\,|\psi; t\rangle.$$

↳ (and so $\psi(x,t)$)

Hence proved $|\psi; t\rangle$ stays a coherent state.

with $\boxed{\alpha(t) = e^{-i\omega t}\alpha}$, $\alpha \in \mathbb{C}$

(b) (a) $\langle x\rangle = \int_{-\infty}^{\infty} dx\, \psi^* x \psi$

$(\Delta x)^2$ ↑width $= \int_{-\infty}^{\infty} dx\, \psi^* (x - \langle x\rangle)^2 \psi$

$= \langle x^2\rangle - \langle x\rangle^2$ It is just variance.

$(\Delta x)^2 = \langle (x - \langle x\rangle)^2 = \langle x^2 - 2x\langle x\rangle + \langle x\rangle^2\rangle$

$= \langle x^2\rangle - \langle x\rangle^2$

Now,

$\langle x\rangle = \langle \psi; t | \hat{x} | \psi; t\rangle$

Now we know $\hat{x} = \sqrt{\frac{\hbar}{2m\omega}} (\hat{a} + \hat{a}^\dagger)$

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Now, we showed.

$$\hat{a}\,|\psi;t\rangle = \alpha(t)\,|\psi;t\rangle$$

$$\therefore \quad \langle\psi;t|\,\hat{a}^\dagger = \alpha^*(t)\,\langle\psi;t|$$

$$\langle x\rangle(t) = \sqrt{\frac{\hbar}{2m\omega}}\,[\alpha(t) + \alpha^*(t)]\,\langle\psi;t|\psi;t\rangle$$

(As we assumed $|\psi\rangle$ is normalised) $\Rightarrow 1$

Let, $\alpha = |\alpha|\, e^{-i\phi}$

$$\therefore \quad \alpha(t) = |\alpha|\, e^{-i(\omega t+\phi)}$$

$$\alpha^*(t) = |\alpha|\, e^{i(\omega t+\phi)}$$

$$\Rightarrow \langle x\rangle(t) = \sqrt{\frac{2\hbar}{m\omega}}\,\big(|\alpha|\,2\cos(\omega t+\phi)\big)$$

$\phi,\ |\alpha| \in \mathbb{R}$

Now, to show that it varies with time, according to classical (Ehrenfest eqn of motion).

i.e., we will show $\boxed{\frac{d}{dt}\langle x\rangle = \frac{\langle p\rangle}{m}}$

We have

$$\frac{d}{dt}\langle x\rangle = \frac{d}{dt}\big(\langle\psi|\,\hat{x}\,|\psi\rangle\big) = \big(\partial_t\langle\psi|\big)\,\hat{x}\,|\psi\rangle + \langle\psi|\,\partial_t\hat{x}\,|\psi\rangle + \langle\psi|\,\hat{x}\,(\partial_t|\psi\rangle)$$

$\Rightarrow 0$.

Considering Schrödinger pict, where operators do not evolve with time.

From SE, $\quad \frac{i}{\hbar}\langle\psi|\,\hat{H}\hat{x}\,|\psi\rangle - \frac{i}{\hbar}\langle\psi|\,\hat{x}\,\hat{H}\,|\psi\rangle$

$$= \frac{i}{\hbar}\langle\psi|\,[\hat{H},\hat{x}]\,|\psi\rangle.$$

$$= \frac{i}{\sqrt{2m\hbar\omega}}\,\langle\psi|\big([\hat{H},\hat{a}^\dagger] + [\hat{H},\hat{a}]\big)\,|\psi\rangle.$$

using $\hat{H} = \left(\hat{a}^\dagger \hat{a} + \frac{1}{2}\right)\hbar\omega$.

$[\hat{a}, \hat{a}^\dagger] = 1$

We find,

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$$= \frac{i}{\sqrt{2\hbar\omega m}} \left(\langle\psi| \left(\hbar\omega \hat{a}^\dagger - \hbar\omega \hat{a}\right) |\psi\rangle \right)$$

$$= \frac{i}{m} \sqrt{\frac{p\,\hbar m\omega}{2}} \left(\langle\psi| \left(\hat{a}^\dagger - \hat{a}\right) |\psi\rangle \right)$$

$$= \frac{i}{m} \langle \hat{p} \rangle$$

Now, similarly

$$\langle x^2 \rangle = \frac{\hbar}{2m\omega} \langle\psi;t| \left((\hat{a}^\dagger)^2 + \hat{a}^2 + \underbrace{\quad\quad}_{2\hat{a}^\dagger\hat{a} + 1} \right) |\psi;t\rangle$$

Calculating

$$= \frac{\hbar}{2m\omega} \left[1 + \left(\alpha^*(t) + \alpha(t)\right)^2 \right]$$

$\because (\Delta x)^2 = \langle x^2 \rangle - \langle x \rangle^2$

$$= \frac{\hbar}{2m\omega} \left[1 + \left(\alpha^*(t) + \alpha(t)\right)^2 - \left(\alpha^*(t) + \alpha(t)\right)^2 \right]$$

$$= \frac{\hbar}{2m\omega}.$$

does not evolve with time.

$\frac{\hbar m\omega}{2} \frac{2}{\hbar m\omega} = 1$

Hence, the width of $\psi(x,t)$ is independent of time!

© For $\langle p \rangle$ = ?

$$\langle p \rangle = \langle\psi;t| \hat{p} |\psi;t\rangle$$

$$= i\sqrt{\frac{\hbar m\omega}{2}} \left(\langle\psi;t| \hat{a}^\dagger - \hat{a} |\psi;t\rangle \right)$$

$$= i\sqrt{\frac{\hbar m\omega}{2}} \left(\alpha^*(t) - \alpha(t) \right)$$

$\Rightarrow$ similarly it evolves with time.

contd. at last page.

and $\frac{-\hbar m\omega}{2}$

$$\langle p^2 \rangle = \left(\langle\psi;t| \left((\hat{a}^\dagger)^2 + (\hat{a})^2 + \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} \right) |\psi;t\rangle \right)$$

$$= \frac{-\hbar m\omega}{2} \left(\langle\psi;t| \left((\hat{a}^\dagger)^2 + \hat{a}^2 - 2\hat{a}^\dagger\hat{a} - 1 \right) |\psi;t\rangle \right)$$

$$= -\frac{\hbar m\omega}{2} \left[\left(\alpha(t) - \alpha^*(t)\right)^2 - 1 \right]$$

$\therefore (\Delta p)^2 = \langle p^2 \rangle - \langle p \rangle^2 = \frac{\hbar m\omega}{2} \left[1 - \left(\alpha - \alpha^*\right)^2 \right]$

Similarly

Also, we will show

Q.3)c) contd. $\frac{d}{dt}\langle p \rangle = -\left\langle \frac{\partial V}{\partial x} \right\rangle$ (Classical Relation)

we have.

$$\frac{d}{dt}\langle p \rangle = \frac{i}{\hbar}\langle \psi | [\hat{H}, \hat{p}] | \psi \rangle$$

$$[\hat{H}, \hat{p}]|\psi\rangle = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V\right)\left(-i\hbar\frac{\partial|\psi\rangle}{\partial x}\right) + \left(i\hbar\frac{\partial}{\partial x}\right)\left(\frac{-\hbar^2}{2m}\frac{\partial^2|\psi\rangle}{\partial x^2} + V|\psi\rangle\right)$$

$$= i\hbar\frac{\partial^3|\psi\rangle}{2m\,\partial x^3} - i\hbar V\frac{\partial|\psi\rangle}{\partial x} - \frac{i\hbar^3}{2m}\frac{\partial^3|\psi\rangle}{\partial x^3} + i\hbar\frac{\partial(V|\psi\rangle)}{\partial x}$$

$$= -i\hbar V\frac{\partial|\psi\rangle}{\partial x} + i\hbar\frac{\partial V}{\partial x} + i\hbar V\frac{\partial|\psi\rangle}{\partial x}$$

Thus $\frac{d}{dt}\langle p \rangle = \frac{i}{\hbar}\, i\hbar \langle \psi | \partial_x V | \psi \rangle$.

$$= -\langle \partial_x V \rangle \quad \square$$

Sumata Chattopadhyay
Roll - 22MS077
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Q.2) $\hat{H} = \frac{e}{m} \bar{S} \cdot \bar{B}$

$\bar{S}$ = spin angular momentum

T.P $\frac{d\vec{S}}{dt} = -\frac{e}{m}(\bar{S} \times \bar{B})$

We have

$\hat{H} = -\gamma \vec{B} \cdot [S]$

$[S]$ - spin matrix

$\gamma = -\frac{e}{m}$ (for e^-).

We have, $[S] = \frac{\hbar}{2}[\sigma]$

when

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$\vec{\sigma} = \sigma_x \hat{x} + \sigma_y \hat{y} + \sigma_z \hat{z}$

$\therefore \hat{H} = \frac{e}{m} \cdot \frac{\hbar}{2} (B_x \sigma_x + B_y \sigma_y + B_z \sigma_z)$

Now from Heisenberg eqn we have,

$\frac{d\vec{S}}{dt} = \frac{i}{\hbar}[\hat{H}, S] = \frac{i}{\hbar}\left(\frac{e}{m}\frac{\hbar}{2}\right)\left(\frac{\hbar}{2}\right)[\vec{B}\cdot\vec{\sigma}, \vec{\sigma}]$

$= \frac{ie\hbar}{4m}[\vec{B}\cdot\vec{\sigma}, \vec{\sigma}]$

Let $\alpha = \frac{ie\hbar}{4m}$

$= \alpha\left[(B_x\sigma_x + B_y\sigma_y + B_z\sigma_z), \sigma_x\hat{i} + \sigma_y\hat{j} + \sigma_z\hat{k}\right]$

Now $[S_i, S_j] = i\hbar\epsilon_{ijk}S_k$

$\frac{\hbar^2}{4}[\sigma_i, \sigma_j] = i\hbar\epsilon_{ijk}\frac{\hbar}{2}\sigma_k$

$$\boxed{[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k}$$

1
+ 2
3

1 – x
2 – y
3 – z

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$= \alpha [B_x \sigma_x, \sigma_x \hat{i}]$

$\equiv \alpha \{ [B_1 \sigma_1, \sigma_1 \hat{i} + \sigma_2 \hat{j} + \sigma_3 \hat{k}]$
$+ [B_2 \sigma_2, \sigma_1 \hat{i} + \sigma_2 \hat{j} + \sigma_3 \hat{k}]$
$+ [B_3 \sigma_3, \sigma_1 \hat{i} + \sigma_2 \hat{j} + \sigma_3 \hat{k}] \}$

$= \alpha \{ 0 + B_1 (2i) \sigma_3 \hat{j} - B_1 (2i) \sigma_2 \hat{k} +$
$- B_2 (2i) \sigma_3 \hat{i} + 0 + B_2 (2i) \sigma_1 \hat{k} +$
$+ B_3 (2i) \sigma_2 \hat{i} - B_3 (2i) \sigma_1 \hat{j} + 0 \}$

$= 2i\alpha \{ \hat{i} (B_3 \sigma_2 - B_2 \sigma_3)$
$+ \hat{j} (B_3 \sigma_1 + B_1 \sigma_3)$
$+ \hat{k} (B_2 \sigma_1 - B_1 \sigma_2) \}$

$= 2i \cdot \frac{ie\hbar}{2\hbar m} \{ \vec{\sigma} \times \vec{B} \}$

$= -\frac{e}{m} \{ \frac{\hbar \vec{\sigma}}{2} \times \vec{B} \}$

$= -\frac{e}{m} (\vec{S} \times \vec{B})$

□.

Q1) For free particle
Post. at $t = 0 \rightarrow x_0$
Mom. at $t = 0 \rightarrow p_0$

(a) we have

$$\frac{d p_x(t)}{dt} = \frac{i}{\hbar} [\hat{H}, \hat{p}_x] = \frac{i}{\hbar} \left[\frac{\hat{p}_x^2}{2m}, \hat{p}_x\right] = 0$$

$$\Rightarrow \frac{d p_x(t)}{dt} = 0 \Rightarrow p_x(t) = p_x(0) = p_0$$

Note technically it is not possible the determin both post. & mom. of a particle simultaneously.

$\therefore \langle p(t) \rangle = p_0$

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Now,

$$\frac{d\hat{x}(t)}{dt} = \frac{i}{\hbar}[\hat{H}, \hat{x}]$$

$$= \frac{i}{\hbar}\left[\frac{\hat{p}_x^2}{2m}, \hat{x}\right] = \frac{i}{\hbar}\frac{1}{2m}\{[\hat{p}, \hat{x}]\hat{p} + \hat{p}[\hat{p}, \hat{x}]\}$$

$$= \frac{i}{2m\hbar}(\hat{p})$$

$$\Rightarrow \frac{d\hat{x}(t)}{dt} = \frac{\hat{p}(t)}{m} = \frac{p_0}{m}$$

Thus, Integrating

$$\hat{x}(t) = \frac{p_0}{m}t + \hat{x}(0)$$

$$\Rightarrow \boxed{\langle \hat{x}(t) \rangle = \frac{p_0 t}{m} + x_0}$$

(b) $\frac{d\hat{p}}{dt} = \frac{i}{\hbar}[\hat{H}, \hat{p}^2]$

From Ehrfest Theorem.

$$\frac{d\langle \hat{p}_x^2 \rangle}{dt} = \frac{i}{\hbar}\langle [\hat{H}, \hat{p}_x^2] \rangle$$

$$= \frac{i}{\hbar}\left\langle \left[\frac{\hat{p}_x^2}{2m}, \hat{p}_x^2\right] \right\rangle$$

$$= 0$$

D

And

$$\frac{d\langle x^2 \rangle}{dt} = \frac{i}{\hbar}\left\langle \left[\frac{\hat{p}_x^2}{2m}, \hat{x}^2\right] \right\rangle$$

$$= \frac{i}{2m\hbar}\langle \hat{x}[\hat{p}_x^2, \hat{x}] + [\hat{p}_x^2, \hat{x}]\hat{x} \rangle$$

$$= \frac{i}{2m\hbar}\langle \hat{x}\hat{p}_x[\hat{p}_x, \hat{x}] + \hat{x}[\hat{p}_x, \hat{x}]\hat{p}_x + \hat{p}_x[\hat{p}_x, \hat{x}]\hat{x} + [\hat{p}_x, \hat{x}]\hat{p}_x\hat{x} \rangle$$

$$= \frac{i\hbar i}{2m\hbar}\langle 2\hat{p}_x\hat{x} + 2\hat{x}\hat{p}_x \rangle$$

Now we use $\hat{x}\hat{p}_x = i\hbar + 2\hat{p}_x\hat{x}$

$$= \frac{1}{2m} \langle 4 \hat{p}_x \hat{x} + 2it \rangle$$

$$\Rightarrow \boxed{\frac{d\langle \hat{x}^2 \rangle}{dt} = \frac{2\langle \hat{p}\hat{x} \rangle}{m} + \frac{it}{m}}$$

□

(c) $(\Delta x)^2 = \langle x^2 \rangle - \langle x \rangle^2$

$$\frac{d\langle \hat{p}\hat{x} \rangle}{dt} = \frac{i}{\hbar}\langle [\hat{H}, \hat{p}_x \hat{x}] \rangle$$

$$= \frac{i}{2m\hbar} \langle [\hat{p}_x^2, \hat{p}_x x] \rangle$$

$$= \frac{i}{2m\hbar} \langle p_x [p_x, p_x x] + [p_x, p_x x] p_x \rangle$$

$[p_x, p_x] = 0$

$$= \frac{i}{2m\hbar} \langle p_x^2 (-i\hbar) - i\hbar p_x^2 \rangle$$

$$= \frac{2i\hbar \cdot i}{2m\hbar} \quad \because \langle \cdot \rangle = \frac{1}{m}\langle p^2 \rangle.$$

Now,

RHS: $\frac{2(\Delta p)^2}{m^2} = \frac{2}{m^2}\left(\langle p^2 \rangle - \langle p \rangle^2\right)$

$$= \frac{2}{m^2}\left(\langle p^2 \rangle - p_0^2\right)$$

$$\langle x \rangle = \frac{p_0 t}{m} + x_0$$

LHS: $\frac{d^2\langle x^2 \rangle}{dt^2} - \frac{d^2\langle x \rangle^2}{dt^2}$

$$= \frac{2}{m}\left(\frac{d}{dt}\langle \hat{p}\hat{x} \rangle\right) + 0 - \frac{2p_0^2}{m^2}$$

$$= \frac{2}{m}\left(\frac{\langle p^2 \rangle}{m}\right) - \frac{2}{m^2}p_0^2$$

$$= \frac{2}{m^2}\left(\langle p^2 \rangle - p_0^2\right) = \text{RHS}$$

□.

Thus $\frac{d^2(\Delta x)^2}{dt^2} = \frac{2(\Delta p)^2}{m^2}$

Q1) ©

we obtained $\frac{d}{dt}\langle p^2 \rangle = 0 \Rightarrow \langle p^2(t) \rangle = \langle p^2(0) \rangle$

Now, given $\frac{d^2\langle \Delta x^2 \rangle^2}{dt^2} = \frac{2}{m^2}(\Delta p)^2$

$$= \frac{2}{m^2}\left(\langle p^2 \rangle - p_0^2\right)$$

$$= \frac{2}{m^2}\left(\langle p^2(0) \rangle - p_0^2\right) = \frac{2}{m^2}(\Delta p_0)^2$$

Given $(\Delta x)^2 = \frac{(\Delta p_0)^2 t^2}{m^2} + (\Delta x_0)^2$

Checking if it's a solution,

$$\frac{d}{dt}\left((\Delta x)^2\right) = \frac{2t(\Delta p_0)^2}{m^2} + 0$$

$$\frac{d^2}{dt^2}\left((\Delta x)^2\right) = \frac{2}{m^2}(\Delta p_0)^2$$

Hence, yes it is!