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Assignment-2 , Group-2
PH3102

Q2) (a) For a real valued $\psi(x)$. T.P $\langle \hat{p} \rangle_\psi = 0$

Since $\psi(x)$ is real valued

$\therefore \quad \psi^*(x) = \psi(x) \quad \forall x$

Now

$$\psi(x) = \langle x|\psi\rangle$$

$$\psi^*(x) = \langle \psi|x\rangle .$$

$$\therefore \langle \hat{p}\rangle_\psi := \langle \psi|\hat{p}|\psi\rangle = \langle\psi| \left(\int_{-\infty}^{\infty} |x\rangle\langle x| dx \right) \hat{p} \left(\int_{-\infty}^{\infty} dx' |x'\rangle\langle x'| \right) |\psi\rangle$$

$$= \int_{-\infty}^{\infty} dx\, dx'\ \psi^*(x) \langle x|\hat{p}|x'\rangle \psi(x')$$

$$= \int_{-\infty}^{\infty} dx\, dx'\ \psi^*(x)\ \hat{p}(x')\ \langle x|x'\rangle\ \psi(x')$$

$$= \int_{-\infty}^{\infty} dx\ \psi^*(x) \left(\int_{-\infty}^{\infty} dx'\ \hat{p}(x')\ \psi(x')\ \delta(x-x') \right)$$

$$= \int_{-\infty}^{\infty} \psi^*(x)\ \hat{p}(x)\ \psi(x)\ .\ dx$$

Now,

let $I = -\int_{-\infty}^{\infty} \psi(x)\,(i\hbar)\,\frac{\partial \psi}{\partial x}\ .\ dx$

applying integration by parts.

$$I = -i\hbar \left[\left(\psi^2(x)\right)\Big|_{-\infty}^{\infty} - \int \left(\frac{\partial \psi}{\partial x}\right) \psi(x)\ .\ dx \right]$$

$\left(\psi^2(x)\right)\Big|_{-\infty}^{\infty} \to 0$ (Sq. integrability/ finiteness of wave function).

$$= -I$$

$$\Rightarrow \quad 2I = 0 \quad \Rightarrow \quad \boxed{I = \langle \hat{p}\rangle_\psi = 0} \qquad \square$$

(b) Now given $\langle \hat{p} \rangle_\psi = \langle \hat{p} \rangle$.

For $\phi(x) = e^{\frac{ip_0 x}{\hbar}} \psi(x)$

$\phi^*(x) = e^{-\frac{ip_0 x}{\hbar}} \psi^*(x)$

$\psi^* = \psi$.

T.P $\langle p \rangle_\phi = \langle \hat{p} \rangle + p_0$.

From previous question, we can write,

$\langle \hat{p} \rangle_\phi := \langle \phi | \hat{p} | \phi \rangle = \int \phi^*(x)\, \hat{p}\, \phi(x)\, dx$

$\therefore = \int dx\, e^{-\frac{ip_0 x}{\hbar}} \psi(x) \cdot -i\hbar \frac{\partial}{\partial x} \left(e^{\frac{ip_0 x}{\hbar}} \psi(x) \right)$

$= -i\hbar \int dx\, e^{-\frac{ip_0 x}{\hbar}} \psi(x) \cdot \left[\frac{ip_0}{\hbar} e^{\frac{ip_0 x}{\hbar}} \psi(x) + e^{\frac{ip_0 x}{\hbar}} \frac{\partial \psi}{\partial x} \right] dx$

$= \cancel{-i\hbar} \int dx\, \psi(x)\, \cancel{e^{-\frac{ip_0 x}{\hbar}}} \cdot \left(\cancel{\frac{ip_0}{\hbar}} \right) \cancel{e^{\frac{ip_0 x}{\hbar}}} \psi(x) + -i\hbar \int dx\, \psi(x) \cdot \cancel{e^{-\frac{ip_0 x}{\hbar}}}\, \cancel{e^{\frac{ip_0 x}{\hbar}}} \frac{\partial \psi}{\partial x}$

$= p_0 \underbrace{\left(\int_{-\infty}^{\infty} dx\, |\psi(x)|^2 \right)}_{= 1} + \underbrace{\int dx\, \psi^*(x) \left(-i\hbar \frac{\partial}{\partial x} \right) \psi}_{= \int dx\, \psi^*(x)\, \hat{p}\, \psi(x)}$

(Total probability of finding particle).

$= \langle \hat{p} \rangle_\psi = \langle \hat{p} \rangle$.

Thus $\langle \hat{p} \rangle_\phi = p_0 + \langle \hat{p} \rangle$.

□

1) $\hat{O} = |\phi\rangle\langle\psi|$

(a) Ansatz: we claim $\hat{O}^\dagger = \alpha|\psi\rangle\langle\phi|$.

consider 2 vectors, $|x\rangle, \langle y|$ of state space. (arbitrary)

we have

$\langle y|\hat{O}x\rangle = \langle y|\phi\rangle\langle\psi|x\rangle$.

- And

$\langle \hat{O}^\dagger y|x\rangle = \left(\langle x|\hat{O}^\dagger y\rangle\right)^\dagger$ (Property of Hermitian inner product)

$= \left(\langle x|\psi\rangle\langle\phi|y\rangle\right)^\dagger$

$= (\langle\phi|y\rangle)^\dagger \left(\langle x|\psi\rangle\right)^\dagger$

$= \langle y|\phi\rangle\langle\psi|x\rangle$.

since $\langle y|\hat{O}x\rangle = \langle \hat{O}^\dagger y|x\rangle$. our claim is verified $\forall |x\rangle, |y\rangle \in \mathcal{H}$.

For $\hat{O}$ to be Hermitian, we require.

$\hat{O} = \hat{O}^\dagger$

$\Rightarrow |\psi\rangle\langle\phi| = |\phi\rangle\langle\psi|$.

$\Rightarrow$ It must be. $|\psi\rangle = \alpha|\phi\rangle$.

where $\alpha \in \mathbb{R}$.

$\therefore \boxed{|\psi\rangle = \alpha|\phi\rangle}$ $\alpha \in \mathbb{R}$.

For Hermiticity of $\hat{O}$

$\Rightarrow$ Now

$|\psi\rangle\langle\phi|\phi\rangle = |\phi\rangle\langle\psi|\phi\rangle$

$|\psi\rangle = \left(\frac{\langle\psi|\phi\rangle}{\langle\phi|\phi\rangle}\right)|\phi\rangle$

when $\alpha \in \mathbb{C}$

$= c|\phi\rangle$

$\therefore \hat{O} = c|\psi\rangle\langle\psi|$

$\hat{O}^\dagger = |\psi\rangle\langle\phi| = c^*|\psi\rangle\langle\psi|$

For Hermitian $\hat{O} = \hat{O}^\dagger$;

$\Leftrightarrow c = c^*$

Thus c must be purely real $c \in \mathbb{R}$

(b) $\hat{O}^2 = \hat{O}\hat{O}$

$= |\phi\rangle\langle\psi|\phi\rangle\langle\psi|$

$= \langle\psi|\phi\rangle\,|\phi\rangle\langle\psi| = \langle\psi|\phi\rangle\,\hat{O}$

$= \langle\psi|\phi\rangle\langle\phi|\psi\rangle\langle\psi|$

Now, for a valid Projection operator ($\hat{O}$), we require $\hat{O} = \hat{O}^2$ (Linear Algebra).

$\Rightarrow \boxed{\langle\psi|\phi\rangle = 1}$.

condition.

(c) T.P $\hat{O} = \lambda P_1 P_2$

$P_1 = |\phi\rangle\langle\phi|$ $\quad \lambda \in \mathbb{C}$

$P_2 = |\psi\rangle\langle\psi|$.

we have.

$\hat{O} = |\phi\rangle\langle\psi|$

c-1. $\langle\phi|\psi\rangle \neq 0$ (i.e they are not orthogonal).

then. $\hat{O} = \frac{1}{\langle\phi|\psi\rangle} \langle\phi|\psi\rangle |\phi\rangle\langle\psi|$.

$= \frac{1}{\langle\phi|\psi\rangle} |\phi\rangle\langle\phi| \; (|\psi\rangle\langle\psi|)$

$= \lambda \cdot P_1 P_2$.

when $\lambda = \frac{1}{\langle\phi|\psi\rangle}$ D

c-2 $\langle\phi|\psi\rangle = 0$. i.e they are orthogonal.

$P_1 P_2 = |\phi\rangle \langle\phi|\psi\rangle \langle\psi|$

$= 0 \neq 0 \cdot \hat{O}$

then. $\hat{O}$ can NOT be written in the form of $\hat{O} = \lambda P_1 P_2$.

Problem 3.

(3) given.

(a) $\langle x|\psi\rangle = \psi(x,0) = \sum_{n=0}^{\infty} c_n \psi_n(x) = \sum_{n=0}^{\infty} \langle x|n\rangle \langle n|\psi\rangle$

($\langle x|n\rangle \to \psi_n(x)$, $\langle n|\psi\rangle \to c_n$)

$\Rightarrow |\psi\rangle = \sum_{n=0}^{\infty} |n\rangle c_n$.

Now, we know $\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle$. (stho).

$\hat{a}|\psi\rangle = \sum_{n=0}^{\infty} \hat{a}|n\rangle c_n$

$= \sum_{n=1}^{\infty} \sqrt{n}\,|n-1\rangle c_n$.

(shifting indices).

$= \sum_{n=0}^{\infty} \sqrt{n+1}\,|n\rangle c_{n+1}$

given $= \alpha|\psi\rangle$

$= \sum_{n=0}^{\infty} \alpha|n\rangle c_n$.

∴ Multiplying by another stat $\langle m|$ of stho both side.

$\Rightarrow \sum_{n=0}^{\infty} \sqrt{n+1}\langle m|n\rangle c_{n+1} = \sum_{n=0}^{\infty} \alpha \langle m|n\rangle c_n$.

Note $\langle m|n\rangle = \delta_{m,n}$

$\therefore \Rightarrow \sqrt{m+1}\, c_{m+1} = \alpha c_m$.

$\Rightarrow c_{k+1} = \frac{\alpha c_k}{\sqrt{k+1}}$ $k = 0, 1, 2, \ldots$

$\Rightarrow c_1$

Now,

$$C_1 = \alpha C_0$$

$$C_2 = \frac{\alpha C_1}{\sqrt{2}} = \frac{\alpha^2 C_0}{\sqrt{2}\cdot\sqrt{1}}$$

$$C_3 = \frac{\alpha C_2}{\sqrt{3}} = \frac{\alpha^3 C_0}{\sqrt{3!}}$$

$\therefore$ Inductively

$$\boxed{C_n = \frac{\alpha^n}{\sqrt{n!}} C_0 \quad n \in \mathbb{N} \cup \{0\}}$$

To find value of C_0, we need to normalise $|\psi\rangle$.

$$\therefore |\psi\rangle = \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} C_0 \; |n\rangle. \qquad C_0 \in \mathbb{C}$$

$$\therefore \langle\psi|\psi\rangle = 1$$

$$\Rightarrow \left(\sum_m \frac{(\alpha^*)^m}{\sqrt{m!}} C_0^* \langle m| \right) \left(\sum_n \frac{\alpha^n}{\sqrt{n!}} C_0 |n\rangle \right) = 1$$

$$= \sum_{m,n} \frac{(\alpha^*)^m \alpha^n}{\sqrt{m!}\sqrt{n!}} |C_0|^2 \langle m|n\rangle = 1.$$

$\langle m|n\rangle$ = 1 orthonormal.

$$\Rightarrow |C_0|^2 \sum_n \frac{|\alpha|^{2n}}{n!} = 1.$$

$$\Rightarrow |C_0|^2 \cdot e^{|\alpha|^2} = 1 \Rightarrow |C_0|^2 = e^{-|\alpha|^2/2}$$

$\therefore$ Required choice of coefficients is

$$\boxed{C_n = \frac{\alpha^n}{\sqrt{n!}} e^{-|\alpha|^2/2}} \quad n \in \mathbb{N} \cup \{0\}.$$

3 (b)

Given $\hat{a}\,\psi(x,0) = \alpha\,\psi(x,0)$.

we know,

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right) = r\left(\hat{x} + \frac{\hbar}{m\omega}\frac{d}{dx}\right).$$

$$r := \sqrt{\frac{m\omega}{2\hbar}}$$

$\therefore$ Putting in eqn. we get.

$$x\psi + \frac{\hbar}{m\omega}\frac{d\psi}{dx} = \frac{\alpha}{r}\psi$$

$$\Rightarrow \frac{\hbar}{m\omega}\frac{d\psi}{dx} = \left(\frac{\alpha}{r} - x\right)\psi \Rightarrow \int \frac{d\psi}{\psi} = \frac{m\omega}{\hbar}\int\left(\frac{\alpha}{r} - x\right)dx$$

$$\Rightarrow \ln\psi = \frac{m\omega}{\hbar}\left(\frac{\alpha}{r}x - \frac{x^2}{2}\right) + C$$

$$\Rightarrow \psi(x,0) = A\exp\left(\frac{m\omega}{\hbar}\left(\frac{\alpha}{r}x - \frac{x^2}{2}\right)\right) \checkmark$$

↳ Needs to be normalised

condition for Normalising is.

$$\int_{-\infty}^{\infty}\psi^*\psi\,dx = 1 \Rightarrow |A|^2\int_{-\infty}^{\infty}\exp\left(\frac{2m\omega}{\hbar}\left(\frac{\tilde{\alpha}}{r}x - \frac{x^2}{2}\right)\right)dx = 1$$

$\tilde{\alpha} = \mathrm{Re}(\alpha)$

$$\Rightarrow |A|^2\int\exp\left(-2r^2\left(x^2 - 2x\left(\frac{\tilde{\alpha}}{r}\right)\right)\right)dx$$

$$+2r^2\left(+\frac{\tilde{\alpha}^2}{r^2}\right)$$

$$\Rightarrow |A|^2 e^{2\tilde{\alpha}^2}\int_{-\infty}^{\infty} e^{-2r^2\left(x - \frac{\tilde{\alpha}}{r}\right)^2}\cdot dx$$

subst. $u = r\left(x - \frac{\tilde{\alpha}}{r}\right)\sqrt{2}$ $\Rightarrow$ $\frac{|A|^2}{r\sqrt{2}}e^{2\tilde{\alpha}^2}\underbrace{\int_{\infty} e^{-u^2}\cdot du}_{=\sqrt{\pi}} = 1$

simplifying, $|A| = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\tilde{\alpha}^2}$

Thus

$$\psi(x,0) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\tilde{\alpha}^2}\exp\left(\frac{m\omega}{\hbar}\left(\frac{\alpha x}{r} - \frac{x^2}{2}\right)\right)$$ Ans.

$\tilde{\alpha} = \mathrm{Re}(\alpha)$, $r = \sqrt{\frac{m\omega}{2\hbar}}$.

(b) Find explicit form for $\psi(x,0)$:

we have

$$|\psi\rangle = \sum_{n=0}^{\infty} \underbrace{\frac{\alpha^n}{\sqrt{n!}} e^{-|\alpha|^2/2}}_{c_n} |n\rangle$$

Now to find $\psi(x,0)$ we have to project it on position basis. i.e, $\psi(x,0) = \langle x|\psi\rangle$

Now:

$$\therefore \psi(x,0) = \sum_{n=0}^{\infty} c_n \langle x|n\rangle = \sum_{n=0}^{\infty} c_n \, a \, \psi_n(x).$$

Now we have

$$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{x} - \frac{i}{m\omega}\hat{p}\right) = \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{x} + \frac{\hbar}{m\omega}\frac{d}{dx}\right)$$

Since $\hat{a}^\dagger |n\rangle = \sqrt{n+1}\,|n+1\rangle$.

we have:

$$\therefore \psi_n(x) = (\hat{a}^\dagger)^n \psi_0(x).$$

where $\psi_0(x) \rightarrow$ ground state wavefunction.

$$\hat{a}|0\rangle = 0.$$

$$\therefore \langle x|\hat{a}|0\rangle = 0 \Rightarrow \langle x|\left(\hat{x} + \frac{\hbar}{m\omega}\left(\hbar i \frac{d}{dx}\right)\right)|0\rangle = 0$$

$$\Rightarrow x\psi_0(x) + \frac{\hbar}{m\omega}\frac{d\psi_0}{dx} = 0$$

$$\Rightarrow \int \frac{d\psi_0}{\psi_0} = -\frac{m\omega}{\hbar}\int x\,dx$$

$$\Rightarrow \psi_0(x) = C\exp\left(-\frac{m\omega}{\hbar}\frac{x^2}{2}\right)$$

After Normalising

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{m\omega}{\hbar}\frac{x^2}{2}\right)$$

This is one one way to find, other way is in P.T.O