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PH 3102 **Assignment - 1**

Q1)(a) (i) $\int_0^\infty x\, e^{-\alpha x^2} \cdot dx$

Ans ~~Solve~~ Take, $u = x^2$

$\rightarrow \frac{du}{2} = x\,dx$

$\therefore I_1 = \frac{1}{2}\int_0^\infty e^{-\alpha u}\, du = \frac{1}{2\alpha}\left[e^{-\alpha u}\right]_0^\infty$

$= \frac{1}{2\alpha}\left[1 - 0\right] = \boxed{\frac{1}{2\alpha}}$

[Requires $\alpha \neq 0$].

Ans.

(ii) $\int_0^\infty x^3\, e^{-\alpha x^2} \cdot dx$

Similarly take, $u = x^2$

$\Rightarrow I_2 = \frac{1}{2}\int_0^\infty u\, e^{-\alpha u} \cdot du$

(By Parts) $= \frac{1}{2}\left\{ -\frac{1}{\alpha}\left(u e^{-\alpha u}\right)\Big|_0^\infty + \frac{1}{\alpha}\int_0^\infty e^{-\alpha t} \cdot dt \right\}$

$= \frac{1}{\alpha}\cdot\frac{1}{2}\int_0^\infty e^{-\alpha t} \cdot dt = \frac{1}{\alpha} I_1 = \boxed{\frac{1}{2\alpha^2}}$

Ans

(iii) $\int_0^\infty x^5\, e^{-\alpha x^2} \cdot dx$.

Take $u = x^2$

$\therefore I_3 = \frac{1}{2}\int_0^\infty u^2 e^{-\alpha u} \cdot du$

$= \frac{1}{2}\left[-\frac{1}{\alpha}\left(u^2 e^{-\alpha u}\right)\Big|_0^\infty + \frac{2}{\alpha}\int_0^\infty u e^{-\alpha u} \cdot du \right]$

$= \frac{2}{\alpha}\left(\frac{1}{2}\int_0^\infty u e^{-\alpha u} \cdot du\right) = \frac{2}{\alpha} I_2$.

$= \frac{2}{\alpha}\cdot\frac{1}{2\alpha^2} = \boxed{\frac{1}{\alpha^3}}$

Ans

$I_3 = \frac{1}{\alpha^3}$.

(b) Take $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$. ($x, y, z$ are all independent variables).

Then in cartesian coordinates.

$\vec{\nabla} \cdot \vec{r}$ =

$$\vec{\nabla} \equiv \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

So, Divergence is.

$$\vec{\nabla} \cdot \vec{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}$$

$$= 1 + 1 + 1 = 3$$

This result is independent of the coordinate system.

$\vec{\nabla} \times \vec{r} =$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \hat{i}\left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right) - \hat{j}\left(\frac{\partial z}{\partial x} - \frac{\partial x}{\partial z}\right) + \hat{k}\left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right)$$

$$= 0.$$

(c) $$\begin{vmatrix} + & y^2+3y & y-1 & y+3 \\ - & y+1 & -2y & y-4 \\ + & y-3 & y+4 & 3y \end{vmatrix} = Ay^4 + By^3 + Cy^2 + Dy + E$$

A, E =?.

$$= (y^2+3y)\left[-7y^2+16\right] + (y-1)\left[y^2-5y+4\right.$$
$$\left. -2y^2-6y\right]$$
$$+ (y+4)\left[2y^2-7y-15\right].$$

Expanding.

$A = -7$

$E = 15 - 12 = 3$

(Q2)



let, The total time taken to fall H = T

$H = \frac{1}{2} g T^2$

$T = \sqrt{\frac{2H}{g}}$ --- (1)

Now, $\dot{x}^2 = 2gx \Rightarrow \dot{x} = \sqrt{2gx}$ (>0)

+ve in downard direction.

$\Rightarrow \frac{dt}{dx} = \frac{1}{\sqrt{2gx}}$ — (2)

From eq (1), (2)

we get $\frac{dt}{T} = \frac{1}{2} \frac{dx}{\sqrt{Hx}}$ probability

Integrating,

$$\int_{t'=0}^{t} \frac{dt'}{T} = \frac{1}{2} \frac{1}{\sqrt{H}} \int_{x'=0}^{x} \frac{dx'}{\sqrt{x'}}$$

$\Rightarrow \frac{t}{T} = \sqrt{\frac{x}{H}} \Rightarrow x = H \left(\frac{t}{T}\right)^2$

So, Average distance fell (from top of RC).

is $\langle x \rangle = \frac{\int_0^T x(t) \cdot dt}{\int_0^t dt} = \frac{\int_0^T \frac{H}{T^2} t^2 \cdot dt}{\frac{1}{T}}$

$= \frac{H}{T^3} \cdot \frac{T^3}{3}$

$= H/3$ Ans.

Q.3)

$$V(x) = \begin{cases} 0 & x < 0 \\ V_0 & x \geq 0 \end{cases}$$

(a) S.E

For $x < 0$. (where ψ is an energy eigenstate)

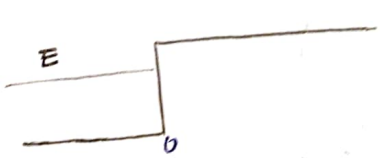
$$\hat{H}\psi = E\psi$$

$$\therefore -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi$$

For $x \geq 0$.

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V_0\psi = E\psi.$$

(b) (i) $E < V_0$



Solving

$$\frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi = -k^2\psi \quad \text{where } k^2 = \frac{2mE}{\hbar^2} > 0.$$

We know,

Solution: $\psi(x) = A_1 e^{ikx} + A_2 e^{-ikx}$ $(x < 0)$.

And for $x \geq 0$.

$$\frac{d^2\psi}{dx^2} = -\frac{2m(E-V_0)}{\hbar^2}\psi = \frac{2m(V_0-E)}{\hbar^2}\psi.$$

Denote $l^2 = \frac{2m(V_0-E)}{\hbar^2} > 0$.

$$\therefore \psi(x) = B_1 e^{lx} + B_2 e^{-lx}.$$

Since, $\psi(x)$ must be square integrable,

At $x \to \infty$, $\psi(x) \to 0$

$\Rightarrow B_1 = 0$.

continuity of ψ (at $x=0$)

$$\Rightarrow \quad A_1 + A_2 = B_2 \quad \cdots (*_1)$$

Differentiability of ψ

$$\Rightarrow \quad ik(A_1 - A_2) = -l B_2 \quad \cdots (*_2)$$
$$= ikl B_2$$

From $(*_1)$ $(*_2)$

$$\Rightarrow \quad 2A_1 = B_2\left(1 + \frac{il}{k}\right)$$

$$\Rightarrow \quad \boxed{B_2 = 2A_1\left(1 + \frac{il}{k}\right)^{-1}}$$

And, $\quad \boxed{A_2 = A_1\left(\frac{k - il}{k + il}\right)}$.

Thus, for $E < V_0$ case.

$$\psi(x) = \begin{cases} A_1 e^{ikx} + A_2 e^{-ikx} & x < 0 \\ B_2 e^{-lx} & x \geq 0 \end{cases}$$

where $\quad l = \frac{1}{\hbar}\sqrt{2m(V_0 - E)}, \quad k = \sqrt{\frac{2mE}{\hbar^2}}$.

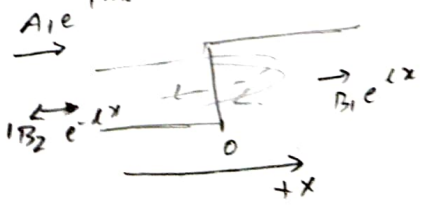
Comment Note that corresponding to any value of E in $0 < E < V_0$, we can get, k & l to vary continuously, and thus get various eigenfunctions.

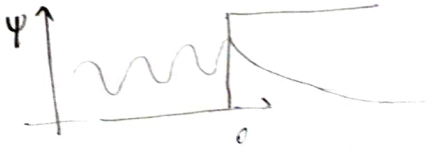
(c) Reflection (R) coefficient

$$R = \left\|\frac{B}{A}\right\|^2 \equiv \left\|\frac{A_2}{A_1}\right\|^2$$

$$R = \left\|\frac{k - il}{k + il}\right\|^2 = \frac{\|k - il\|^2}{\|k + il\|^2}$$

$$= \left(\frac{k^2 + l^2}{k^2 + l^2}\right) = 1.$$





Transmission coefficient – It is the probability that a particle will continue right on through.

$$T := \frac{l}{K}\left\|\frac{B_1}{A_1}\right\|^2$$

clearly as $B_1 = 0$.

$\therefore T = 0$

A_1 = amplitude of incident wave
A_2 = " " reflected wave.
B_1 = " " transmitted wave.

And so $R + T = 1$ (As was expected)

Comment: We see that, even at $x \geq 0$, clasically we ~~do not expect~~ no particle should have existed in this region, however in quantum mechanical sense $\psi(x) \neq 0$ at $x \geq 0$, which indicates a non-zero probability of finding particle in this region. This is called quantum tunnelling.

And $R = 1$, indicates that although the wave function penetrates into the barrier, it is eventually all reflected back.

(d) $P(x) = \psi\psi^* = |\psi|^2$

For $x \geq 0$

we have. $\psi(x) = B_2 e^{-lx}$.

$$\therefore |\psi|^2 = |B_2|^2 e^{-2lx} = \frac{4|A_1|^2}{\left|1 + \frac{il}{K}\right|^2} e^{-2lx} = \frac{4|A_1|^2}{1 + \frac{l^2}{K^2}} e^{-2lx}.$$

$\therefore$

$$P(x) = \frac{4K^2}{K^2 + l^2}|A_1|^2 e^{-2lx} \quad \text{for } x \geq 0.$$

For $x \to 0$ $\quad P(x) = \frac{4K^2}{K^2+l^2}|A_1|^2 = \frac{4x\ \frac{2mE}{\hbar^2}}{\frac{2m}{\hbar^2}V_0}|A_1|^2 = \frac{4E}{V_0}|A_1|^2 \neq 0$

clearly

$$P(x)\Big|_{x\to\infty} = 0 \quad (\because e^{-2\ell x} \to 0 \text{ as } x \to \infty).$$

∴ we calculated:

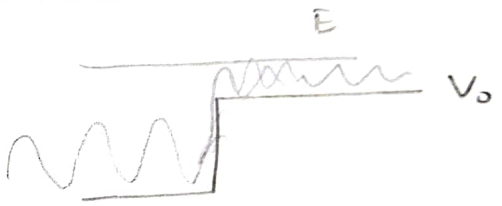
$$P(x) = \frac{4E}{V_0}|A_1|^2 e^{-2\ell x}, \quad \forall x \geq 0$$

$$|A_1|^2 \in \mathbb{R}^+$$

$$P(x)\Big|_{x\to\infty} = 0$$

$$P(x)\Big|_{x\to 0} = \frac{4E}{V_0}|A_1|^2 > 0.$$

(b) (i) $E > V_0$



The $x < 0$ part is same as before

only for $x \geq 0$ we have:

$$\frac{d^2\psi}{dx^2} = -\frac{2m(E-V_0)}{\hbar^2}\psi = -k'^2\psi(x).$$

$$k'^2 = \frac{2m(E-V_0)}{\hbar^2}.$$

By assumption we neglect a contribution $De^{-ik'x}$ corresponding to an incoming wave from the $x > 0$ region.

So, we have:

$$\psi(x) = \begin{cases} Ae^{ikx} + Be^{-ikx}, & x < 0 \\ Fe^{ik'x}, & x \geq 0. \end{cases}$$

comment: Similarly it is also a free particle in all the region, so energy can be allowed to vary continuously. (Also, note the wavefunctions are NOT integrable ⇒ they are not proper solutions, and thus are not found in Hilbert space).

(c) To determine A, B, F.

we solve the boundary conditions at $x = 0$.

$$\psi(0^+) = \psi(0^-)$$

$$\psi'(0^+) = \psi'(0^-)$$

$$A + B = F$$

$$ikF(A - B) = ik'F$$

$$\Rightarrow A + B = \frac{k'}{k}(A - B).$$

$$\Rightarrow A\left(1 - \frac{k}{k'}\right) = -B\left(1 + \frac{k}{k'}\right).$$

$$\therefore \Rightarrow$$

Reflecting coefficient

$$R = \left|\frac{B}{A}\right|^2 = \left|\frac{1 - \frac{k}{k'}}{1 + \frac{k}{k'}}\right|^2 = \frac{(k' - k)^2}{(k' + k)^2}$$

Now.

$$F = A + B = A + A\left(\frac{1 - \frac{k}{k'}}{1 + \frac{k}{k'}}\right)$$

$$= A\left[\frac{1 + \frac{k}{k'} - 1 + \frac{k}{k'}}{1 + \frac{k}{k'}}\right]$$

$$\Rightarrow \frac{F}{A} = \frac{2(k/k')}{1 + k/k'}.$$

$$\therefore$$

Transmittance coefficient,

$$T = \frac{k'}{k}\left|\frac{F}{A}\right|^2 = \frac{4(k/k')^2}{(1 + k/k')^2} \cdot \frac{k'}{k}$$

$$= \frac{4k/k'}{(1 + k/k')^2}$$

check that

$$R + T = \left(\frac{1 - k/k'}{1 + k/k'}\right)^2 + \left(\frac{4\, k/k'}{(1 + k/k')^2}\right)$$

$$= \frac{(1 + k/k')^2}{(1 + k/k')^2} = 1 .$$

Calculating Analytic Expressions

~~Note~~

$$\left(\frac{k'}{k}\right)^2 = \frac{2m(E - V_0)/\hbar^2}{2mE/\hbar^2} = 1 - \frac{V_0}{E} .$$

$$R = \frac{(k' - k)^2}{(k' + k)^2} = \frac{(k' - k)^4}{\left((k')^2 - (k)^2\right)^2}$$

Now

$$k^2 - k'^2 = \frac{2m}{\hbar^2}(E - E + V_0) = \frac{2m}{\hbar^2} V_0 .$$

$$k - k' = \frac{\sqrt{2m}}{\hbar}\left[\sqrt{E} - \sqrt{E - V_0}\right] .$$

So, $R = \dfrac{\left(\frac{2m}{\hbar^2}\right)^2 \left(\sqrt{E} - \sqrt{E-V_0}\right)^4}{\left(\frac{2m}{\hbar^2}\right)^2 V_0^2}$

$$\therefore \boxed{R = \frac{\left(\sqrt{E} - \sqrt{E-V_0}\right)^4}{V_0^2}}$$

2 Transmittance coefficient

$$T = \frac{4kk'}{(k+k')^2} = \frac{4kk'(k-k')^2}{\left(k^2 - (k')^2\right)^2}$$

$$= 4\left(\frac{2m}{\hbar^2}\right)^2 \frac{\sqrt{E}\sqrt{E-V_0}\left(\sqrt{E} - \sqrt{E-V_0}\right)^2}{V_0^2 \left(\frac{2m}{\hbar^2}\right)^2}$$

$$\boxed{T = \frac{4\sqrt{E}\sqrt{E-V_0}\left(\sqrt{E} - \sqrt{E-V_0}\right)^2}{V_0^2}}$$

Comment: The fact that $E > V(x)$ in both the regions makes it a scattering state, so same as before ⊛ Allowed energy states can vary continuously.