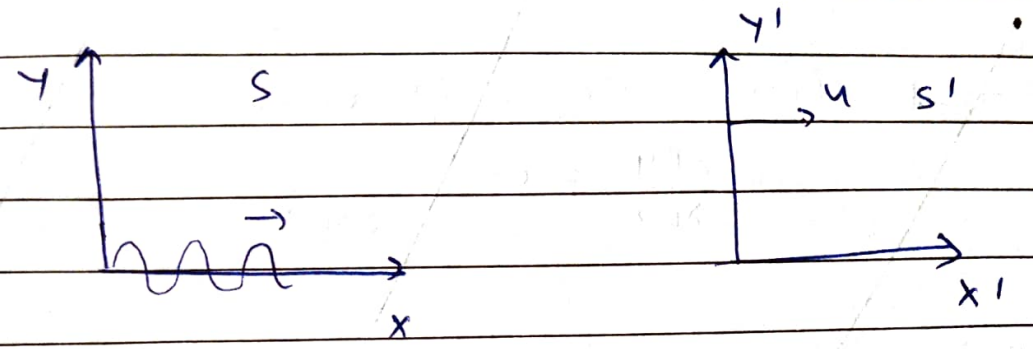
(3) consider $\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}$

Let us consider the wave propagating along x-axis with speed 'v' in S frame.

We would like to find it's eqn in frame S', which is comoving ~~with~~ with S frame along x axis with velocity u.



The Lorentz Transformation eqns are.

$x' = \gamma(x - ut) = \gamma(x - \beta ct)$ $\quad \frac{\partial}{\partial t} = \frac{\partial}{\partial x'} \frac{\partial x'}{\partial t}$

$y' = y$

$z' = z$

$t' = \gamma\left(t - \frac{ux}{c^2}\right) = \gamma\left(t - \frac{\beta x}{c}\right)$

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}$$

$$\Rightarrow \frac{\partial}{\partial t}\left(\frac{\partial y}{\partial t}\right) = v^2 \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right)$$

$$\frac{\partial}{\partial t} \equiv \frac{\partial}{\partial x'} \cdot \frac{\partial x'}{\partial t} + \frac{\partial}{\partial t'} \cdot \frac{\partial t'}{\partial t}$$

$$= -\beta\gamma c \frac{\partial}{\partial x'} + \gamma \frac{\partial}{\partial t'}$$

Similarly,

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x'} \cdot \frac{\partial x'}{\partial x} + \frac{\partial}{\partial t'} \cdot \frac{\partial t'}{\partial x}$$

$$= \gamma \frac{\partial}{\partial x'} + -\frac{\beta\gamma}{c} \frac{\partial}{\partial t'}$$

$$\therefore \frac{\partial}{\partial t}\left(\frac{\partial}{\partial t}\right) \equiv \left(-\beta\gamma c \frac{\partial}{\partial x'} + \gamma \frac{\partial}{\partial t'}\right)^2$$

$$= \beta^2\gamma^2 c^2 \frac{\partial^2}{\partial x'^2} + \gamma^2 \frac{\partial^2}{\partial t'^2} - 2\beta\gamma^2 c \frac{\partial^2}{\partial x' \partial t'}$$

And,

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\right) = \left(\gamma \frac{\partial}{\partial x'} - \frac{\beta\gamma}{c} \frac{\partial}{\partial t'}\right)^2$$

$$= \gamma^2 \frac{\partial^2}{\partial x'^2} + \frac{\beta^2\gamma^2}{c^2} \frac{\partial}{\partial t'^2} - \frac{2\beta\gamma^2}{c} \frac{\partial^2}{\partial x' \partial t'}$$

Writing $v^2 = \beta^2 c^2$

Equating both sides.

$$\beta^2\gamma^2 c^2 \frac{\partial^2 y}{\partial x'^2} + \gamma^2 \frac{\partial^2 y}{\partial t'^2} - 2\beta\gamma^2 c \frac{\partial^2 y}{\partial x' \partial t'}$$

$$= v^2 \left[\gamma^2 \frac{\partial^2 y}{\partial x'^2} + \frac{\beta^2\gamma^2}{c^2} \frac{\partial^2 y}{\partial t'^2} - \frac{2\beta\gamma^2}{c} \frac{\partial^2 y}{\partial x' \partial t'}\right]$$

$$\frac{\partial^2 y}{\partial t'^2}\left(\gamma^2 - \frac{v^2\beta^2\gamma^2}{c^2}\right)$$

$$= \frac{\partial^2 y}{\partial x'^2}\left(\gamma^2 v^2 - \beta^2\gamma^2 c^2\right) + \frac{\partial^2 y}{\partial x' \partial t'}\left(2\beta\gamma^2 c - v^2 \frac{2\beta\gamma}{c}\right)$$

Clearly the eqn is not Lorentz invariant.

It is invariant only when the term involving mixed partial derivative is zero

ie $2\beta\gamma^2 c = v^2 \frac{2\beta\gamma}{c}$

$$v = \pm c.$$

$$\therefore \frac{\partial^2 y'}{\partial t'^2}\left(\gamma^2 - \beta^2\gamma^2\right) = c^2 \frac{\partial^2 y'}{\partial x'^2}\left(\gamma^2 - \beta^2\gamma^2\right)$$

$$\therefore \left(\frac{\partial^2 y'}{\partial t'^2} - c^2 \frac{\partial^2 y'}{\partial x'^2}\right)\left(\gamma^2 - \beta^2\gamma^2\right) = 0$$

$$\Rightarrow \boxed{\frac{\partial^2 y'}{\partial t'^2} = c^2 \frac{\partial^2 y'}{\partial x'^2}}$$

And $\gamma^2 - \beta^2\gamma^2 = 0$ which is true.

So, this is also verified by the fact that speed of light is invariant in all comoving inertial frames.

(3)

Checking Invariancy under Galilean Transformation.

$x' = x - ut$.

$y' = y$

$z' = z$

$t' = t$.

Given,

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}.$$

$$\therefore \frac{\partial}{\partial t} \equiv \frac{\partial}{\partial x'}\frac{\partial x'}{\partial t} + \frac{\partial}{\partial t'}\frac{\partial t'}{\partial t}.$$

$$= -u\frac{\partial}{\partial x'} + \frac{\partial}{\partial t'}$$

$$\therefore \frac{\partial^2}{\partial t^2} = u^2\frac{\partial^2}{(\partial x')^2} + \frac{\partial^2}{(\partial t')^2} - 2u\frac{\partial^2}{\partial x'\partial t'}$$

Now $\frac{\partial}{\partial x} = \frac{\partial}{\partial t'}\frac{\partial t'}{\partial x} + \frac{\partial}{\partial x'}\frac{\partial x'}{\partial x}$ ($\frac{\partial t'}{\partial x} = 0$)

$= \frac{\partial}{\partial x'}$

$$\therefore \frac{\partial^2}{\partial x^2} = \frac{\partial^2}{(\partial x')^2}$$

$$\left(\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial t}\right) y$$

$$= v^2\left\{\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\right)\right\} y.$$

(since $y = y'$)

$\therefore$ Putting.

$$u^2 \frac{\partial^2 y'}{\partial x'^2} + \frac{\partial^2 y'}{\partial t'^2} - \frac{2u\partial^2 y}{\partial x' \partial t'} = v^2 \frac{\partial^2 y'}{\partial x'^2}.$$

$\Rightarrow$

$$-\frac{\partial^2 y'}{\partial t'^2} = \frac{\partial^2 y'}{\partial x'^2}(v^2 - u^2) + \frac{2u\partial^2 y'}{\partial x' \partial t'}$$

$\therefore$ Eqn is not Galilean Invariant.

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① Given a dispersion relation, $\omega = v K$.

we construct a wavepacket by choosing sinusoids of form $\sin(Kx - \omega t)$ from $-K_0/4$ to $K_0/4$. with uniform amplitude for any K.

a) Calculate the shape of wave packet.
b) What is group velocity of packet.

A We want to obtain, all possible sinusoid relation.
(K = wave number)

So, Given Amplitude is uniform, so,

~~$y = \int A_1$~~ A = constant.

Eq^n of wave packet.

$$y = \int_{-K_0/4}^{K_0/4} A e^{i(Kx - \omega t)} \cdot dK$$

=

$$y = A \int_{-K_0/4}^{K_0/4} e^{i(Kx - vKt)} \cdot dK.$$

$$= A \cdot \int_{-K_0/4}^{K_0/4} e^{iK(x - vt)} \cdot dK.$$

$$y = \frac{A}{i(x-vt)} \left[e^{ik(x-vt)} \right]_{-K_0/4}^{K_0/4}$$

$$y = \frac{A}{i(x-vt)} \left[e^{\frac{iK_0(x-vt)}{4}} - e^{-\frac{iK_0}{4}(x-vt)} \right]$$

Let $x - vt = h$.

$$\therefore y = \frac{A}{hi} \left[e^{\frac{iK_0 h}{4}} - e^{-\frac{iK_0 h}{4}} \right]$$

~~$= f(h)$~~.

~~So this represents a travelling wave.~~

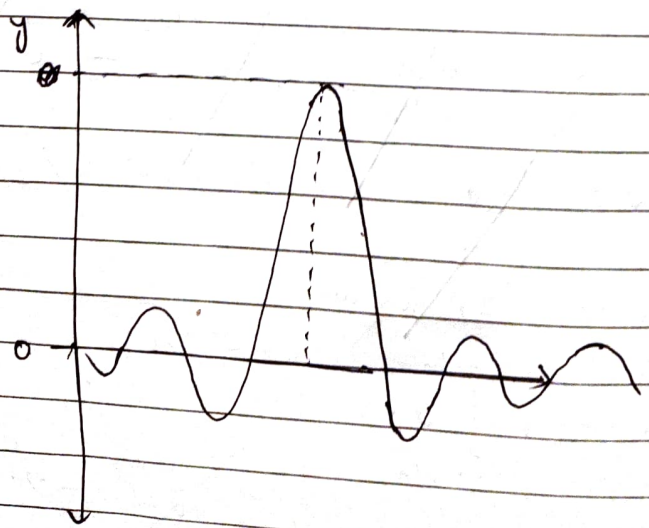
$$= \frac{A}{hi} \left[\times 2i \sin\left(\frac{K_0 h}{4}\right) \right]$$

$$y = \frac{2A}{x-vt} \left[\sin\left(\frac{K_0}{4}(x-vt)\right) \right]$$

This the equation for the travelling wave.

- The wave packet will look at some time like.

At

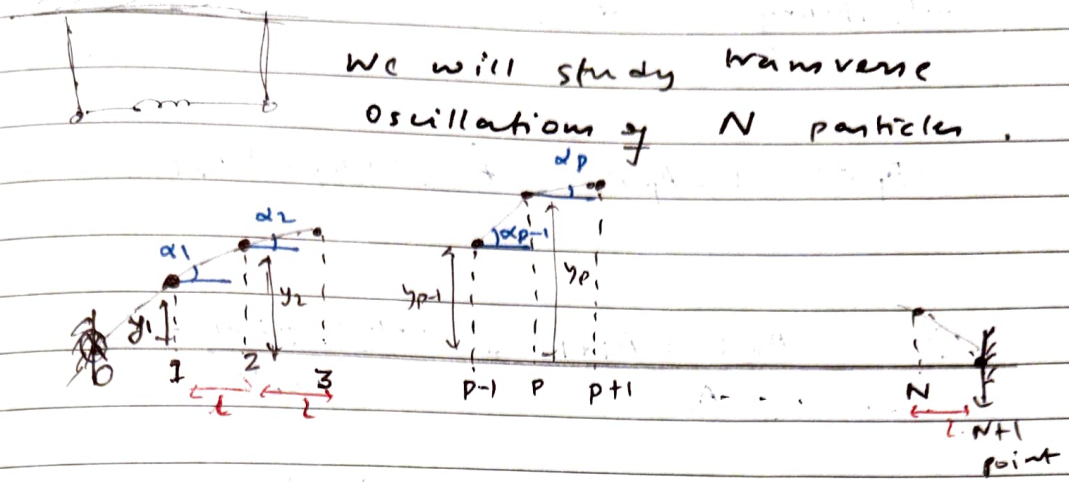


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b) Here, group velocity = phase velocity,

$$v_g = \frac{d\omega}{dk} = v = \frac{d\omega}{dK}.$$

(4)



We write the force acting on p^{th} particle.

$$F = -T \sin\alpha_{p-1} + T \sin\alpha_p$$

considering small angle approximation, we have

$$m\ddot{y}_p = F \simeq -T\left(\frac{y_p - y_{p-1}}{l}\right) + T\left(\frac{y_{p+1} - y_p}{l}\right)$$

$$\Rightarrow \ddot{y}_p = -\frac{T}{ml}\left[2y_p - y_{p-1} - y_{p+1}\right].$$

Take $\frac{T}{ml} = \omega_0^2$

$$\frac{d^2}{dt^2} y_p + 2\omega_0^2 y_p - \omega_0^2 (y_{p-1} + y_{p+1}) = 0$$

Note that

$p = 0$, $p = N+1$ points are stationary.

We seek to find the Normal modes, where each particle have the same frequency ω.

Let, $y_p = A_p \cos\omega t$.

$\dot{y}_p = -A_p \omega \sin\omega t \rightarrow \dot{y}_p(0) = 0$.

[$\because$ Each particle starts from rest].

$\ddot{y}_p = -A_p \omega^2 \cos\omega t$.

Substituting values.

$$-A_p\omega^2 + 2A_p\omega_0^2 - \omega_0^2(A_{p-1} + A_{p+1}) = 0$$

$$\Rightarrow \quad \frac{A_{p-1} + A_{p+1}}{A_p} = \frac{-\omega^2 + 2\omega_0^2}{\omega_0^2}$$

$$p = 1, 2, 3 \ldots N$$

We see, for any value of ω, RHS is constant.

Values of A_p = ? तो 2θ मध्ये घेतो

RHS = constant & $A_0 = A_{N+1} = 0$

Try:

$$A_p = C \sin p\theta$$

$$\therefore \quad \frac{[\sin(p-1)\theta + \sin(p+1)\theta]}{\sin p\theta}$$

$$= \frac{[\sin p\theta\cos\theta - \cos p\theta\sin\theta + \sin p\theta\cos\theta + \cos p\theta\sin\theta]}{\sin p\theta}$$

$$\therefore \quad 2\cos\theta = \frac{-\omega^2 + 2\omega_0^2}{\omega_0^2} \rightarrow \text{independent of } p \quad \text{— (1)}$$

(6)

This putting Boundary conditi-

Now, $A_p = C \sin p\theta$.

$\therefore A_0 = 0$.

at $p = N+1$.

$\Rightarrow |\sin p\theta| = \pm 1$ (At Amplitude is Maximum at Antinode)

$\Rightarrow$ ie, $\sin(N+1)\theta = \pm 1$

$\Rightarrow (N+1)\theta = m\pi + \pi/2$.

$\theta = \frac{m\pi}{N+1} + \frac{\pi}{2(N+1)}$

$\therefore A_p^{(m)} = C \sin\left(\frac{p\pi}{N+1}\left(m+\frac{1}{2}\right)\right)$

Note at $(N+1)$: $l = L$

$\therefore$ under continuum limit, $pa = x$

↑ eqlbm post of each particle

$$A_p^{(m)} = C_m \sin\left(\frac{(m+1/2)\pi x}{L}\right)$$

$$= C_m \sin(Kx)$$

where $K = \frac{(m+1/2)\pi}{L}$.

Further, if we put $A_p = C \sin p\theta$.

Then

$$\frac{-\omega^2 + 2\omega_0^2}{\omega_0^2} = 2\cos\theta.$$

$$\Rightarrow \omega = 2\omega_0 \sin\frac{\theta}{2} \qquad \left(\omega_0 = \sqrt{\frac{T}{md}}\right)$$

$$\Rightarrow \omega = 2\omega_0 \sin\left(\frac{1}{2}\left(\frac{(m+1/2)\pi}{N+1}\right)\right)$$

$$= 2\omega_0 \sin\left(\frac{Kd}{2}\right).$$

For N very large.

$$\omega \simeq 2\omega_0 \cdot \frac{KL}{2} = \sqrt{\frac{T}{mL}} \cdot KL.$$

$$= K \cdot \sqrt{\frac{T}{m/l}} \cdot l \qquad \text{Let } \mu = m/l$$

$$= K\sqrt{\frac{T}{\mu}}$$ Linear Mass Density.

where $\omega_m = vK$.

which indeed agrees with velocity of wave in a string.

So, the general equation is

$$y(x,t) = \sum_{m=0}^{\infty} C_m \sin\left[\frac{(m+1/2)\pi x}{L}\right] \cos\left(\frac{(m+1/2)\pi v t}{L} + \phi\right)$$

We will obtain the wavelengths in those (normal) modes, where we get an antinode at end of string,

ie, For ie,

$|\sin KL| = 1$ $\quad$ for $m=0$, for $m=1$

ie, $\frac{2\pi}{\lambda} \cdot L = \frac{\pi}{2}, \frac{3\pi}{2}, 5\pi/2, \ldots$

$\Rightarrow \lambda = 4L, 4L/3, 4L/5, \ldots$

(2) Consider. $\ddot{x} + \alpha\dot{x} + \omega_0^2 x = 0$

(a) Find solution, for given $x(0) = x_0$ and $\dot{x}(0) = v_0$

(b) To find the solution for critically damped case, check the solution at the limit of vanishing resonance frequency.

A (a) Let $x = Ae^{i\omega t}$.

$\dot{x} = Ai\omega e^{i\omega t}$

$\ddot{x} = -A\omega^2 e^{i\omega t}$.

Putting in eq.

$$(-\omega^2 + \alpha i\omega + \omega_0^2) Ae^{i\omega t} = 0$$

$$\Rightarrow \omega = \frac{\alpha i \pm \sqrt{(\alpha i)^2 + 4\omega_0^2}}{2}$$

$$\Rightarrow \omega = \frac{i\alpha}{2} \pm \omega_1$$

where $\omega_1 = \sqrt{\omega_0^2 - \frac{\alpha^2}{4}}$.

$$\therefore x(t) = Ae^{it\left(\frac{\alpha i}{2} + \omega_1\right)} + Be^{it\left(\frac{\alpha i}{2} - \omega_1\right)}$$

$$x(t) = e^{-\alpha t/2}\left[Ae^{i\omega_1 t} + Be^{-i\omega_1 t}\right]$$

$t = 0$,

$$x_0 = A + B.$$

$$\dot{x}(0) = v_0.$$

$$\frac{-iv_0}{\omega_1} = A - B$$

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Adding and subtracting.

$$A = \frac{1}{2}\left[x_0 - i\frac{v_0}{\omega_1}\right].$$

$$B = \frac{1}{2}\left[x_0 + i\frac{v_0}{\omega_1}\right].$$

$$\therefore x(t) = \frac{e^{-\alpha t/2}}{2}\left[\left(x_0 - i\frac{v_0}{\omega_1}\right)e^{i\omega_1 t} + \left(x_0 + i\frac{v_0}{\omega_1}\right)e^{-i\omega_1 t}\right]$$

using

$$e^{i\omega_1 t} = \cos(\omega_1 t) + i\sin(\omega_1 t)$$

we obtain,

$$x(t) = e^{-\alpha t/2}\left[x_0\cos(\omega_1 t) + \frac{v_0}{\omega_1}\sin(\omega_1 t)\right]$$

where, $\omega_1 = \sqrt{\omega_0^2 - \frac{\alpha^2}{4}}$

This is fine as long as $\omega_0 > \frac{\alpha}{2}$ (underdamped)

What happens if $\omega_0 < \frac{\alpha}{2}$, we investigate this.

Then $\omega_0^2 - \frac{\alpha^2}{4} < 0$

$\therefore$ $\omega_1 \rightarrow$ becomes imaginary

$\therefore$ $i\omega_1$ = Real

$\Rightarrow$ we replace ω_1 with $i\omega_1$

$$\therefore x(t) = e^{-\alpha t/2}\left[x_0\cos(i\omega_1 t) + \frac{v_0}{\omega_1}\sin(i\omega_1 t)\right]$$

From De Moivre Theorem, we know.

$$e^{i\omega_1 t} = \cos\omega_1 t + i\sin\omega_1 t$$

$$\therefore e^{-i\omega_1 t} = \cos(i\omega_1 t) + i\sin(i\omega_1 t).$$

$\omega_1 \rightarrow i\omega_1$ & $e^{i\omega_1 t} = \cos(i\omega_1 t) - i\sin(i\omega_1 t)$.

Adding

$$\cos(i\omega_1 t) = \frac{e^{i\omega_1 t} + e^{-i\omega_1 t}}{2}, \quad \sin(i\omega_1 t) = \frac{(e^{-i\omega_1 t} - e^{i\omega_1 t})}{2i}$$

$$\therefore x(t) = \frac{x_0}{2}\left(e^{\omega_1 t} + e^{-\omega_1 t}\right) + \frac{v_0}{i2\omega_1}\left(e^{-\omega_1 t} - e^{\omega_1 t}\right)$$

$$x(t) = e^{-\alpha t/2}\left\{e^{\omega_1 t}\left(\frac{x_0}{2} + i\frac{v_0}{2\omega_1}\right) + e^{-\omega_1 t}\left(\frac{x_0}{2} - i\frac{v_0}{2\omega_1}\right)\right\}$$

overdamped.

Note that RHS is real, since ω_1 is imaginary

$\therefore \frac{i}{\omega_1}$ = Real.

(b) At critically damped, $\omega_0 \approx \frac{\alpha}{4}$

$\therefore \omega_1 \to 0$

$$\therefore \lim_{\omega_1 \to 0} x(t) = \lim_{\omega_1 \to 0}\left[x_0 \cos\omega_1 t + \frac{v_0 t}{\omega_1 t}\sin\omega_1 t\right]$$

$$\Rightarrow x(t) = (x_0 + v_0 t)\, e^{-\alpha t/2}$$

