No. 47473

IISER KOLKATA

भारतीय विज्ञान शिक्षा एवं अनुसंधान संस्थान कोलकाता
INDIAN INSTITUTE OF SCIENCE EDUCATION AND RESEARCH KOLKATA

S.B. 8/12

Invigilator's Signature

To be filled by the student : Please tick (✓) appropriate boxes below:

Mid-Semester Examination		End-Semester Examination	✓	Autumn Semester	✓	Spring Semester	

Roll No.		2	2	M	S	0	7	7	Semester No.	3	Name	SUSNATA CHATTOPADHYAY
Subject No.		M	A	2	1	0	2		Subject Name	Linear Algebra 1.		
									Name of the Parent Department			

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1. Fill out the details of examination and your Name, Roll No. etc. on this cover page, supplementary sheets and on the bottom right hand corner of graph sheets, if used.
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TO BE FILLED IN BY THE EXAMINER

Question No.	1	2	3	4	5	6	7	8	9	10	Total	Signature of Examiner With date
Marks Obtained	2+6	8	0+9	5+9	4						27+5	

Signature of Scrutineer (with date)

Marks Obtained (in words) = 32 + 9 = 41

Full Marks: 50

LINEAR ALGEBRA I (MA2102)

MID-SEM EXAMINATION

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TEACHING ASSISTANTS: MD. NURUL MOLLA, RAJIV MISHRA, ABISEK DEWAN

Exercise 1. (8 points)
Let V be a finite dimensional vector space and let W_1 and W_2 be two subspaces of V. Show that

$$W_1^0 + W_2^0 = (W_1 \cap W_2)^0.$$

Exercise 2. (3+7 points)
Let $P_2(t)$ denote the space of polynomials in t with real coefficients and of degree less than or equal to 2. Consider the symmetric bilinear form $s : P_2(t) \times P_2(t) \to \mathbb{R}$ given by

$$s(f, g) = f(0) \cdot g(0) + f(1) \cdot g(1) + f(2) \cdot g(2).$$

a) Show that the form s is positive definite.
b) Find an orthonormal basis of $P_2(t)$ with respect to the form s.

Exercise 3. (6+7 points)
a) Let A be a 2×2 nondiagonalizable complex matrix of trace 4. Find the trace of A^{2023}.
b) Let B be an $n \times n$ complex matrix such that $B^2 = B$. Show that that the trace of B is the same as the rank of B.

Exercise 4. (5+9 points)
a) Let A be an $n \times n$ real matrix and let $S = \{v_1, v_2, \ldots, v_n\}$ be a set of eigenvectors of A such that $v_i \perp v_j$ (with respect to dot product) if $i \neq j$. Show that A is symmetric.
b) Let $T : \mathbb{C}^n \to \mathbb{C}^n$ be a normal operator and let v and v' be two eigenvectors corresponding to two distinct eigenvalues of T. Show that $v \perp v'$ with respect to the standard Hermitian product.

Exercise 5. (4+8 points)
a) Let A be an $n \times n$ complex matrix and let $B = A - \overline{A}^{\mathrm{T}}$. Show that B is diagonalizable and the real parts of all the eigenvalues of B are 0.
b) Let M be a $n \times n$ Hermitian matrix such that $M^{1000} = I$. Show that $M^3 - 2M^2 - M + 2I = 0$.

① To prove $W_1^\circ + W_2^\circ = (W_1 \cap W_2)^\circ$

$W_1^\circ + W_2^\circ = \{ f \in V^* : f = f_1 + f_2 \text{ s.t. } f_i \in W_i^\circ \}$

$(W_1 \cap W_2)^\circ = \{ f \in V^* : f(u) = 0 \ \forall u \in W_1 \cap W_2 \}$

Part 1

Consider $f \in W_1^\circ + W_2^\circ$

Then $f = f_1 + f_2$ where $f_i \in W_i^\circ$

Let $u \in W_1 \cap W_2$

$\Rightarrow u \in W_1$ and $u \in W_2$.

$\therefore f(u) = f_1(u) + f_2(u) = 0 + 0$

This is true for any arbitrary vector in $W_1 \cap W_2$.

Hence, $f \in (W_1 \cap W_2)^\circ$

$\therefore W_1^\circ + W_2^\circ \subseteq (W_1 \cap W_2)^\circ$.

Part 2

We consider a basis for $W_1 \cap W_2$

$(\alpha_1, \dots \alpha_k)$

Extend it to basis of $W_1 \rightarrow (\alpha_1, \dots \alpha_k, \beta_1, \dots \beta_l)$

and basis of $W_2 \rightarrow (\alpha_1, \dots \alpha_k, \gamma_1, \dots \gamma_m)$.

This can be further extended to the basis of V

$$B = (\alpha_1, \dots \alpha_k, \beta_1, \dots \beta_l, \gamma_1, \dots \gamma_m, \lambda_1, \dots \lambda_n)$$

Then any $v \in V$ can be written as →

$$v = \sum_{i \le k} x_i \alpha_i + \sum_{i \le l} y_i \beta_i + \sum_{i \le m} z_i \gamma_i + \sum_{i \le n} w_i \lambda_i$$

Take any $f \in V^*$

$f(\alpha_i) = a_i$, $f(\beta_i) = b_i$, $f(\gamma_i) = c_i$, $f(\lambda_i) = d_i$

Then

$$f(v) = \sum_{i \le k} x_i a_i + \sum_{i \le l} y_i b_i + \sum_{i \le m} z_i c_i + \sum_{i \le n} w_i d_i$$

$x_i, y_i, z_i, w_i \in \mathbb{R}$

For some real a_i, b_i, c_i, d_i

Now, if $f \in (W_1 \cap W_2)^0$

then for any $v \in W_1 \cap W_2$.

~~$w_i = 0 \ \forall i$~~

$\because$ basis

$$\therefore v = \sum_{i \le k} x_i \alpha_i + \sum_{i \le l} y_i \beta_i + \sum_{i \le m} z_i \gamma_i$$

and $f(v) = \sum_{i \le k} x_i a_i + \sum y_i b_i + \sum_{i \le m} z_i c_i$

$= 0$ ... (1)

$f(\alpha_i) = a_i$

We define $f = f_1 + f_2$.

We define $f = f_1 + f_2$.

$(\ ) \ge 0$

Again :- $v \in W_1 \cap W_2$

$v \in W_1$ can also be written as $v \in W_1^\circ$

It can also be written as

$v = \sum_{i \le k} x_i a_i + \sum_{i \le l} y_i \beta_i + \sum_{i \le m} z_i c_i$ ... (1)

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$f(v) = 0$ from eq (1).

$\Rightarrow f(w) = \sum_{i \le l} y_i \beta_i + \sum_{i \le m} z_i c_i = 0$

Define, $f = f_1 + f_2$ s.t.

- $f_1(v) = \sum_{i \le m} z_i c_i = 0 \quad \forall v \in W_1$ $\because (z_i = 0) \therefore f_1 \in W_1^\circ$
- $f_2(v) = \sum_{i \le l} y_i \beta_i = 0 \quad \forall v \in W_2$ $(\because \sum y_i = 0) \therefore f_2 \in W_2^\circ$

Hence any $f \in (W_1 \cap W_2)^\circ$ can be expressed as $f = f_1 + f_2$ $(f_i \in W_i^\circ)$ $\therefore (W_1 \cap W_2)^\circ \subseteq W_1^\circ + W_2^\circ$

Hence, from part 1 & part 2.

$(W_1 \cap W_2)^\circ = W_1^\circ + W_2^\circ$

D.

Ex-2 $S(f,g) = f(0) \cdot g(0) + f(1) \cdot g(1) + f(2) \cdot g(2)$.

(a) Let, $f = ax^2 + bx + c, \quad a, b, c \in \mathbb{R}$.

$\neq 0$.

$S(f,f) = f^2(0) + f^2(1) + f^2(2)$.

$= c^2 + (a+b+c)^2 + (4a+2b+c)^2$

clearly, as $a, b, c \in \mathbb{R}$ and not all a, b, c are equal to zero → why?

$\Rightarrow S(f,f) > 0 \quad \forall f \in P_2(t)$ $(f \neq 0)$

Thus the form s is positive definite.

(b) consider basis

$B = \{u_1, u_2, u_3\} = \{1, x, x^2\}$ of $P_2(t)$

We shall find an orthonormal basis (ONB) of $P_2(t)$ by Gram Schmidt orthogonalisation process.

Let $v_1 = u_1 = 1$ $\quad \therefore w_k = \frac{v_k}{\|v_k\|}$

Note: $\|v_k\| = \sqrt{s(v_k, v_k)}$

$\therefore w_1 = \frac{1}{\sqrt{3}}$

$v_2 = u_2 - \frac{\langle u_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1$

$= x - \frac{s(x,1)}{s(1,1)} \times 1$

$= x - \frac{3}{3} \times 1$

$= x - 1$

$\therefore w_2 = \frac{v_2}{\|v_2\|} = \frac{x-1}{\sqrt{2}}$

And,

$v_3 = u_3 - \frac{\langle u_3, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2 - \frac{\langle u_3, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1$

$= x^2 - \frac{s(x^2, x-1)}{2} - \frac{s(x^2, 1)}{3} \times 1$

Rough.

$s(f,g) = f(0)g(0) + f(1)\cdot g(1) + f(2)\cdot g(2)$

$= 1 + 1 + 1$

$= 0 + 1 + 2$

$= 3$

$s(x-1, x-1)$

$= \sqrt{(-1)(-1) + 0 + 1 \cdot 1}$

$= +1 + 1$

$= \sqrt{2}$

$s(1,1)$

$s(x^2, x-1)$

$= 0 + 0 + 4(1)$

$= 4$

$0 + 1 + 4$

$$v_3 = x^2 - \frac{x^2}{2}(x-1) - \frac{5}{3} \cdot 1$$

$$= x^2 - 2x + 2 - \frac{5}{3}$$

$$= x^2 - 2x + \frac{1}{3}$$

$$\therefore w_3 = \frac{v_3}{\|v_3\|} = \frac{\left(x^2 - 2x + \frac{1}{3}\right)}{\sqrt{5\left(x^2-2x+\frac{1}{3},\ x^2-2x+\frac{1}{3}\right)}}$$

$$= \sqrt{\frac{3}{2}}\left(x^2 - 2x + \frac{1}{3}\right)$$

$5\left(x^2-2x+\frac{1}{3},\ x^2-2x+\frac{1}{3}\right)$

$$= \left(\frac{1}{3}\right)^2 + \left(\frac{3}{3} + \frac{1}{3}\right)^2 + \left(x - x + \frac{1}{3}\right)^2$$

$$\left(-\frac{2}{3}\right)^2$$

$$= \frac{2}{9} + \frac{4}{9} = \frac{6}{9} = \frac{2}{3}$$

So, the ONB is,

$$\{w_1, w_2, w_3\} = \left\{\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{2}}(x-1), \sqrt{\frac{3}{2}}\left(x^2-2x+\frac{1}{3}\right)\right\}$$

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(a) $A \in M_n(\mathbb{R})$.

~~consider~~ $S = \{v_1, v_2, \dots v_n\}$.

$\because v_i \perp v_j \quad \forall i \neq j \Rightarrow$ S is a linearly independent set.

Let $c_1 \dots c_n \in \mathbb{R}$.

$c_1 v_1 + c_2 v_2 + \dots c_n v_n = 0$.

We dot product with v_i

$\Rightarrow c_1 v_1^2 = 0 \Rightarrow c_1 = 0 \quad (\because v_1 \neq 0)$

In this way we can show

$c_1 = \dots = c_n = 0$

Ex-4

(a) Let A be the mx. of some linear operator $T: V \to V$ w.r.t. $\{v_1, \dots v_n\}$ are the eigenvectors of T, and $S = \{v_1, \dots v_n\}$ is orthogonal.

They are linearly independent

∴ We can construct an ONB of V.

$$B = \{w_1, \dots w_n\}$$

$$w_i = \frac{v_i}{\|v_i\|}$$

Let, D be the gram's matrix of form $\langle , \rangle : V \times V \to \mathbb{R}$ w.r.t basis B.

clearly $D = I$.

Let, v be an eigenvector with eigen value = λ. $v = [B]X$

Then, $\langle v, Tv \rangle = \langle v, \lambda v \rangle = \lambda \langle v, v \rangle$

$= X^T (AX)$

Also, $\langle Tv, v \rangle = \langle \lambda v, v \rangle$

$= \lambda \langle v, v \rangle$

equating.

$(AX)^T X = X^T (AX)$

$\Rightarrow X^T A^T X = X^T A X$

$\Rightarrow A = A^T$ Hence, A is symmetric.

Ex-5

(a) $B = A - A^*$

B is

$B^* = (A - A^*)^*$

$= A^* - A$

$BB^* = (A - A^*)(A^* - A)$

$= AA^* - A^2 - (A^*)^2 + A^*A$

$B^*B = (A^* - A)(A - A^*)$

$= A^*A - (A^*)^2 - A^2 + AA^*$

$\therefore BB^* = B^*B$

$\therefore$ B is a Normal Matrix

By Spectral theorem, there exist an orthonormal basis of V, consisting of eigen vectors of B.

P.T.O 2nd part

$\Rightarrow$ Eigen basis exist.

(Theorem in class) $\Leftarrow$ B is diagonalizable.

Hence proved D.

5 (b) M = hermitian Matrix.

By spectral Theorem, $\exists\ P \in U_n$ s.t

$P^*MP = D$. (D consisting of eigenvalues of D).

$\therefore M = P^*DP$

$M^2 = P^*DP\ P^*DP = P^*D^2P$

$M^{1000} = P^*D^{1000}P = I$.

$PP^* = I$

$\therefore P^* = P^{-1}$

$M = M^*$

(a) $\therefore \quad P^* D^{1000} P = I$

$\Rightarrow \quad P^* D^{1000} P = P^{-1}$

$\Rightarrow \quad P P^* D^{1000} = P P^{-1}$

$\Rightarrow \quad D^{1000} = I.$

$\Rightarrow$

$M^3 - 2M^2 - M + 2I = 0$

$M^2(M-I) - M(M-I) - 2(M-I) = 0$

$(M-I)(M^2 - M - 2) = 0$

$(M-I)(M-2I)(M+I) = 0$

Since M is an Hermitian matrix then we know all eigenvalues of M are real.

$\Rightarrow$ D is a real matrix. (consist of real entry)

So, $D = I$

$\begin{pmatrix} \lambda_1^{1000} & & \\ & \ddots & \\ & & \lambda_n^{1000} \end{pmatrix} = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$

$\Rightarrow \lambda_i^{1000} = 1$

$\Rightarrow \lambda_i = \pm 1 \quad \forall i$ (i ∈ N)

$\therefore \quad D = I$ ✗ ✓ $(P \in U_n)$

Hence, $M = P^* P$

So, $M^3 - 2M^2 - M + 2I$

$P^* D^3 P - 2(P^* D^2 P) - (P^* D P) + 2I$

$\Rightarrow P^* P - 2 P^* P - P^* P + 2I = 0$

$I - 2I - I + 2I = 0$

D.

5 (a) 2nd part

Let v be an eigenvector of B, associated with eigenvalue λ

$\because$ B is Normal, Then

$\therefore$ v is also eigenvector of B^* with eigenvalue, $\bar{\lambda}$.

Observe that

$$B^* = -B$$

$$\therefore B^* v = -Bv$$

$$\Rightarrow \bar{\lambda} v = -\lambda v$$

$$\Rightarrow (\bar{\lambda} + \lambda) v = 0$$

$\because v \neq 0$.

$$\Rightarrow \bar{\lambda} = -\lambda$$

Let $\lambda = a + ib$

$$a - ib = -a - ib.$$

$$\Rightarrow a = 0.$$

$\Rightarrow \lambda$ is purely Real.

since this true for any arbitrary eigenvalue

$\Rightarrow$ All eigenvalues of A is have real part $= 0$.

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Exerci

4) (b)

$TT^* = T^*T$ (Normal)

Let, $Tv = \lambda v$ $(\lambda \neq \lambda')$

$Tv' = \lambda' v'$

To show $\langle v, v' \rangle = 0$

As, T is Normal we know,

by spectral Theorem,

we can get an orthonormal basis of V, which consists of eigenvectors of T.

$\therefore S = \{v_1, \ldots v_n\}$ are eigenvectors a linearly independent, and also orthogonal to each other.

So, for any two distinct eigen values of T. v, v' will be belong to different eigenspaces which are independent.

$(V = E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_k})$

Clearly, $\langle v, v' \rangle = 0$

$\therefore$ i.e v is $\perp$ v'

□