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L.Alg. Assignment 2

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Q) Find the dimension of $\text{Span}_{\mathbb{R}} S$, where

$$S := \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix} \right\}.$$

Ans $\text{Span } S = \left\{ \alpha \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \gamma \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix} : \alpha, \beta, \gamma \in \mathbb{R} \right\}$

~~We check that if the set of vectors in Span S are linearly independent.~~

We check that if $\exists$ any linear relation among the vectors $\{v_1, v_2, v_3\} := S$

Take, $c_1, c_2, c_3 \in \mathbb{R}$

if, $c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$

$$\Rightarrow \begin{pmatrix} c_1 + c_3 \\ c_1 + c_2 + 4c_3 \\ c_2 + 3c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow$ ~~$c_1 = -c_3 \; ; \; c_2 = -3c_3$~~

Check that, $c_3 = 1$, $c_2 = -3$, $c_1 = -1$ is a solution of the system of eqn.

$\Rightarrow \exists$ a non-trivial solution

$\Rightarrow \exists$ a linear relation among the vectors

$\Rightarrow \{v_1, v_2, v_3\}$ are not lin. independent.

Now, take any $\bar{v} \in \text{Span } S$.

Then $\bar{v} = \alpha \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \gamma \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}$ $\alpha, \beta, \gamma \in \mathbb{R}$

$$= \alpha \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \gamma \left[\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 3 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right]$$

$$= (\alpha + \gamma) \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + (\beta + 3\gamma) \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

2.

$$\bar{v} = c_1\begin{pmatrix}1\\1\\0\end{pmatrix} + c_2\begin{pmatrix}0\\1\\1\end{pmatrix} \quad \text{------} \circledast$$

where $c_1, c_2 \in \mathbb{R}$.

Claim: $\mathcal{B} = \left\{\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}\right\}$ is basis of span S.

Take, $a, b \in \mathbb{R}$.

If,

$$a\begin{pmatrix}1\\1\\0\end{pmatrix} + b\begin{pmatrix}0\\1\\1\end{pmatrix} = 0$$

$\Rightarrow a = 0 = b$. $\Rightarrow$ The set of vectors is linearly independent.

Further, Note that from $\circledast$ span $\mathcal{B}$ = span S

$\therefore$ This proves our claim

□

Hence,

dim span S = $|\mathcal{B}| = 2$

Ans

Ex 2

(i) Let $A = \begin{bmatrix}1 & 1 & 1\\ 2 & 2 & 3\\ x & y & z\end{bmatrix}$, $V = \{(x,y,z) \in \mathbb{R}^3 : |A| = 0\}$.

Show V is vector space & find dim V. = ?

Ans $\det A = 0 \Rightarrow x = y$

Claim 1 $(V, +)$ is a Group.

(i) Associativity.

Take $\bar{v}, \bar{u}, \bar{w} \in V$, $\bar{v} = (x_0, x_0, z_0) \in \mathbb{R}^3$

$\bar{u} = (x_1, x_1, z_1) \in \mathbb{R}^3$

$\bar{w} = (x_2, x_2, z_2) \in \mathbb{R}^3$.

Then,

$\bar{v} + (\bar{u} + \bar{w}) =$ ~~$(x_1+x_2, x_1+x_2, z_1+z_2) = (x_2+x_1, x_2+x_1, z_1+z_2)$~~

$(x_0, x_0, z_0) + (x_1+x_2, x_1+x_2, z_1+z_2)$ ~~$= (x_2, x_2, z_2) + (x_1+x_1, z_1)$~~

$= (x_0+x_1+x_2,\ y_0+y_1+y_2,\ z_0+z_1+z_2)$

$= (x_0+x_1, y_0+y_1, z_0+z_1) + (x_2, y_2, z_2)$ Date / Page /

$= (\bar{v}+\bar{u})+\bar{w}$ □

~~Claim 2 ... (iii)~~

(ii) Existance of identity element:

Note that $\bar{0} = (0,0,0) \in \mathbb{R}^3 \in V$,

s.t. $\bar{u}+\bar{0} = (x_1+0, y_1+0, z_1+0) = \bar{u}$

(iii) Existance of additive inverse.

choose, $\bar{u} = (x_1, y_1, z_1)$ } $\in \mathbb{R}^3$

$\bar{u}' = (-x_1, -y_1, -z_1)$

$\therefore \bar{u}+\bar{u}' = (x_1-x_1, y_1-y_1, z_1-z_1) = (0,0,0) = i_a \in V$

$\therefore (V,+)$ is a group. □

Claim 2 $(V,+)$ is abelian.

observe that,

$\bar{u}+\bar{w} = (x_1+x_2, y_1+y_2, z_1+z_2)$ | $+$ is commutative over field $\mathbb{R}$ |

$= (x_2+x_1, y_2+y_1, z_2+z_1)$

$= \bar{w}+\bar{u}$ □

Claim 3 Existance of scalar multiple $\mathbb{R}\times V \to V$

Take $\alpha \in \mathbb{R}$, $\bar{v} \in V$. αx_0 $\mathbb{R}^3$

$\therefore \Rightarrow \alpha(x_0, y_0, z_0) = (\alpha x_0, \alpha y_0, \alpha z_0) \in \mathbb{R}^3$

$\Rightarrow$

$\therefore \alpha\bar{v} \in V$.

From Claim 1,2,3, it is proved that V is a vector space. □

(3) Let A be an $n\times n$ real matrix such that sum of entries of each column of A is 1. Show that 1 is an eigenvalue of A.

Claim 1: A and A^T have same eigenvalues.

Proof:

We know, eigenvalues are the root of charac. eqn.

$|A-\lambda I| = 0$.

Now, $|A^T-\lambda I| = |(A-\lambda I)^T| = |A-\lambda I|$

$\therefore$ Eq^{n}s $|A^T-\lambda I|$ & $|A-\lambda I|$ have the same roots.

□

Claim 2: λ is eigenvalue of A.

If we are able to show that, λ is an eigenvalue of A^T, then we will prove our claim.

Take, $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & & \\ \vdots & & & \\ a_{n1} & \cdots & & a_{nn} \end{bmatrix}_{n \times n}$ $\quad A \in M_n(\mathbb{R})$.

Given. $\sum_{k=1}^{n} a_{kj} = \lambda \quad \forall j \in \mathbb{N}$.

(b)

$A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & & & \\ \vdots & & & \\ a_{1n} & \cdots & & a_{nn} \end{bmatrix}_{n \times n}$.

Take. $\bar{v} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \in \mathbb{R}^n$.

$\therefore A^T v = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & \cdots & & \vdots \\ \vdots & \cdots & & a_{nn} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} \sum_{k \le n} a_{k1} \\ \sum_{k \le n} a_{k2} \\ \vdots \\ \sum_{k \le n} a_{kn} \end{bmatrix}$

$= \begin{bmatrix} \lambda \\ \lambda \\ \vdots \\ \lambda \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \lambda \bar{v}$

Thus ~~λ is~~ .

~~it~~ v is an eigenvector of A^T

Corr, λ is an eigenvalue of A^T

$\Rightarrow \lambda$ is an " of A

$\square$.

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Q) 2)i)

$$V = \{(x_1 - x_2 + x_3,\ x_1 + x_2 - x_3) : (x_1, x_2, x_3) \in \mathbb{R}^3\}$$

Show that V is a vector space & Find dim V.

We have

$$V = \{(x_1 - (x_2 - x_3),\ x_1 + (x_2 - x_3)) : (x_1, x_2, x_3) \in \mathbb{R}^3\}$$

$$= \{(x_1 - x_1'),\ (x_1 + x_1') : (x_1, x_1') \in \mathbb{R}^2\}$$

Since $x_2 - x_3$ can be any arbitrary real no.

$$= \{x_1(1,1) + x_1'(-1,1) : x_1', x_1 \in \mathbb{R}\}$$

Since, span of any set of vectors in $\mathbb{R}^2$, forms a vector space

$\Rightarrow$ V = vector space. D

Now, $B = \{(1,1), (-1,1)\}$

Take, $c_1, c_2 \in \mathbb{R}$.

Then $c_1(1,1) + c_2(-1,1) = (0,0)$

$\Rightarrow c_1 = c_2 = 0$.

$\Rightarrow$ B is a linearly independent set.

Also span (B) = V.

$\Rightarrow$ dim V = $|B| = 2$.

Q.4) For a prime p, Find the cardinality of $GL_n(\mathbb{F}_p)$.

A Take $A = GL_n(\mathbb{F}_p)$

Then

$A = [v_1 \ldots v_n]$.

where $v_i \in \mathbb{F}_p^n$

$\mathbb{F}_p = \{\bar{0}, \bar{1} \ldots \overline{p-1}\}$.

Since A is invertible.

$\therefore \exists\ A^{-1}$ s.t.

$$AA^t = I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & 0 \\ 0 & 0 & & 0 \\ 0 & \vdots & & 0 \\ \vdots & \vdots & & 1 \end{bmatrix}$$

$$[v_1 \cdots v_n] A^t = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & 1 \end{bmatrix}$$

Observe that, a linear combinations of columns of A are giving rise to each columns in $I_n \in \mathbb{F}_p^n$.

Thus, ~~$[v_1 \cdots v_n]$ correspond~~
$(v_1, \cdots v_n)$ corresponds to basis of $\mathbb{F}_p^n$.
↓
Hypervector

$\Rightarrow$ $v_1 \cdots v_n$ must be linearly independent

Now,

$$v_1 = \begin{pmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{pmatrix}$$

For each element there 'p' choices, but we need to eliminate the case when $v_1 = \bar{0}$.

$\because$ $\det A \neq 0$,

$\therefore$ Total possible $v_1 = p^n - 1$

Now, for $v_2 \rightarrow$ it must not be a scalar multiple of v_1

There are 'p' scalars $\in \mathbb{F}_p$

$\therefore$ Ways for $v_2 = p^n - p$.

similarly v_k $\forall$ $1 \leq k \leq n$,
must not be a scalar multiple of prev. vectors, ie. $v_1, v_2 \cdots v_{k-1}$

Thus,
$\therefore$ Ways for $v_k = p^n - p^{k-1}$

Thus, There would be a total of
$(p^n - 1)(p^n - p) \cdots (p^n - p^{n-1})$ distinct bases of $\mathbb{F}_p$. ~~So $|GL_n(\mathbb{F}_p)| =$~~

So,
$|GL_n(\mathbb{F}_p)| = \prod_{j=0}^{n-1} (p^n - p^j)$

A

(5) Let V be a finite dimensional real vector space, and let $W_1, W_2, \ldots W_n$ be proper subspaces of V.

Show that $V \neq \bigcup_{i=1}^{n} W_i$

Ans We remove subspaces which satisfy

$W_{k_1}, W_{k_2} \ldots \subseteq \bigcup_{\substack{i=1 \\ i \neq k_1, k_2 \ldots}}^{n} W_i$

(finitely many)

Since $\bigcup_{i \in I} W_i = \left(W_{k_1} \cup W_{k_2} \ldots\right) \cup \bigcup_{\substack{i=1 \\ i \neq k_1, k_2 \ldots}}^{n} W_i = \bigcup_{\substack{i=1 \\ i \neq k_1, k_2, \ldots}}^{n} W_i$

We index subspaces again.

$I := \{1, \ldots m\}$ as $W_i' \; \forall \, i \in I$.

Clearly $\bigcup_{i=1}^{n} W_i = \bigcup_{\substack{i=1 \\ i \neq k_1, k_2 \ldots}}^{n} W_i = \ldots \bigcup_{i \in I} W_i'$

We wish to show that

$V \neq \bigcup_{i \in I} W_i'$.

On the contrary, Let $V = \bigcup_{i \in I} W_i'$

Now by construction of W_i',

$\exists \, u \in W_1'$ s.t. $u \notin \bigcup_{i \in I \setminus \{1\}} W_i'$

$\Rightarrow u \notin W_i' \; \forall \, i \in I \setminus \{1\}$ ----(1)

Again as $W_1' \subset V$ (Proper subset).

$\therefore \exists \, v \in V$ s.t. $v \notin W_1'$ ------(2).

$\therefore v \in \bigcup_{i \in I} W_i' \setminus W_1'$

So, $u, v \in \bigcup_{i \in I} W_i'$

$\alpha v + u$

$\Rightarrow \ldots \in \bigcup_{i \in I} W_i'$ (As it is vector space)

$\alpha \in \mathbb{R} \; (\neq 0)$

If

$\exists\ \alpha v + u \in W_i'$

$\therefore \frac{1}{\alpha}\{-w + \alpha v + u\} = v \in W_i'$ (∵ It is a subspace)

which is a contradiction acc. to eq (2)

$\Rightarrow u + \alpha v \in V \setminus W_i'$

By Pigeonhole principle, $\exists$ some W_ℓ', $\ell \geq 1$ s.t, it contains infinitely many vectors of form, $u + \alpha v$.

Let W_ℓ' contains $u + \alpha_1 v$ & $u + \alpha_2 v$ $(\alpha_1 \neq \alpha_2)$

So, $\frac{\{(u+\alpha_2 v) - (u+\alpha_1 v)\}}{\alpha_2 - \alpha_1} = v \in W_\ell'$

And $(u + \alpha_1 v) + (-\alpha_1 v) = u \in W_\ell'$

Since, u was an arbitrary nonzero vector in W_i'

we see $W_i' \subseteq \bigcup_{i \in I} W_i' \setminus W_i'$ So, $V = \bigcup_{i \in I \setminus \{1\}} W_i'$

continuing this process, we get many

$$V = \bigcup_{i=1}^{n} W_i = \bigcup_{i \in I} W_i' = \bigcup_{i \in I \setminus \{1\}} W_i' = \bigcup_{i \in I \setminus \{1,2\}} W_i'$$

$$= \cdots = W_n' \subset V$$

which is a contradiction

$\therefore V \neq \bigcup_{i=1}^{n} W_i$

□