Date
Page

Q.1) (i) Explain why a system of homogenous linear equations with more variables than equations always has a non-trivial solution, whereas a system of such equations with more equations than variable may not have any non-trivial solution.

Ans Consider a set of linear homogenous equation

$$a_{11}x_1 + a_{12}x_2 + \cdots a_{1m}x_m = 0$$
$$a_{21}x_1 + a_{22}x_2 + \cdots a_{2m}x_m = 0$$
$$\vdots \qquad\qquad \vdots$$
$$a_{n1}x_1 + \cdots + a_{nm}x_m = 0$$

Here

n :- No. of equations. (Rows)

m :- No. of variables. (Columns).

Case 1 $m > n$

consider

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & & a_{nm} \end{bmatrix}_{n\times m} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}_{m\times 1} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n\times 1}$$

Let us denote the matrix by $A = [A]_{n\times m}$

$\therefore \quad AX = 0$

We note that, if A and B are $n\times m$ matrices over field F, we say that B is row-equivalent to A if B can be obtained from A by a finite sequence of elementary row operation.

We assume that, A is row-equivalent to a row-reduced echelon matrix R, because $\exists$ elementary row operations s.t.

$$E_1 E_2 \cdots E_p A = R. \quad \cdots\cdots (*)$$

Then the system

$$AX = 0$$

and $RX = 0$ have the same solution.

(as the rows of R are actually linear combination of rows of A).

If r = No. of non-zero rows in R, then certainly $r \leq n$

and since $n < m$

we have, $r < m$

Then always we can choose some x_j which is not among the r unknowns $(x_{k_1}, \cdots x_{k_r})$ and we can construct a non-zero solution of x_j

i.e., $\begin{pmatrix} x_{k_1} \\ \vdots \\ x_{k_r} \end{pmatrix} \in F^r - \{0\}$

for some x_j

$j \in \{k_1, \cdots k_r\}$

This follows immediately that system will always have a non-trivial solution. $\square$

Case 2 $m < n$.

We have no. of rows $>$ no. of columns.

Let r be number of non-zero rows in R.

certainly $r \leq n$.

but $\because n > m$.

We cannot guarantee that $r < m$.

If k_i be the position of the first non-zero element of the i^{th} row for $1 \leq i \leq r$.

Let u_j be the j^{th} element of X, such that it's position $p \notin \{k_1, \ldots k_r\}$.

If $r \geq m$, then, no u_j might exist.

(i.e. we could land up in redundant rows, where $\exists$ a linear relation among them, or non-intersecting hyper planes.)

So, a Non-trivial solution might not exist.

For eg:

$$\left.\begin{aligned} 3x + 5y &= 0 \\ 5x + 3y &= 0 \\ x + y &= 0 \end{aligned}\right\}$$ has only solution $(x, y) = (0, 0)$

□.

(ii) Show that a matrix with more columns than rows (or rows > columns) does not have a left (or right) inverse.

Let $A_{m \times n}$ matrix over the field F.

If $\exists$ a matrix $B_{n \times m}$ s.t,

(i) $BA = I$, then B is called left inverse of A

(ii) $AB = I$, then B is called right inverse of A.

Given, $n > m$.

Suppose, $B_{n \times m}$ is a left inverse of A.

Then $BA = I_n$.

We note that the row vectors of I_n span F^n. (and also they form basis of F^n).

Again, ∵ each row of I_n is actually linear combinations of rows of A.

⇒ rows of A also span F^n (∵ m rows)

∴ Set $\{A_1, \ldots A_m\}$ span F^n.

⇒ $m \geq n$.

which is a contradiction, since $m < n$.

∴ B cannot exist.

~~B = A~~

So, $A_{m \times n}$ cannot have a left inverse, if $m < n$.

□

Now, let us consider case when $m > n$,

⇒ No. of rows > No. of columns.

Let $B_{n \times m}$ be right inverse of A.

Then

$$AB = I_n \Rightarrow (AB)^T = I_n^T = I_n.$$

$$B^T A^T = I.$$

Which contradicts our previous result, since A^T (has more columns than rows), and consequently cannot have a left inverse.

∴ $A_{m \times n}$ cannot have a right inverse if $m > n$

Exn Show that an $n \times n$ real matrix is invertible, iff columns span $\mathbb{R}^n$.

Pt 1 Columns span $\mathbb{R}^n \Rightarrow A$ is invertible ($A \in M_n(\mathbb{R})$.)

Ans Let $C_1, C_2, \ldots C_n$ are columns of A.

Given Columns span $\mathbb{R}^n$

$\Rightarrow$ $\{C_1, C_2, \ldots C_n\}$ is linearly independent and forms a basis of $\mathbb{R}^n$.

If E_i, $i = 1, \ldots n$ are standard basis for $\mathbb{R}^n$, we can say

$$E_i = C_1 B_{1i} + C_2 B_{2i} + \cdots + C_n B_{ni} \quad \text{— (i)}$$

for a unique ordered pair $(B_{1i}, B_{2i}, \ldots B_{ni})$

Using the standard basis of $\mathbb{R}^n$, and elements A_{ij} of matrix A, we know any column can be written as

$$C_i = A_{1i} E_1 + A_{2i} E_2 + \cdots + A_{ni} E_n \quad \text{— (ii)}$$

Substituting (ii) in (i),

$$E_i = (A_{11} E_1 + \cdots + A_{n1} E_n) B_{1i} + (A_{12} E_1 + \cdots + A_{n2} E_n) B_{2i} + \cdots + (A_{1n} E_1 + \cdots + A_{nn} E_n) B_{ni}$$

$$E_i = (A_{11} B_{1i} + \cdots + A_{1n} B_{ni}) E_1 + \cdots + (A_{i1} B_{1i} + \cdots + A_{in} B_{ni}) E_i + \cdots + (A_{n1} B_{1i} + \cdots + A_{nn} B_{ni}) E_n.$$

Comparing RHS & LHS, it is evident that

$$A_{i1} B_{1i} + \cdots + A_{in} B_{ni} = 1 \quad \forall i = 1, \ldots n. \quad \text{— (iii)}$$

whereas

$$(A_{j1} B_{1i} + \cdots + A_{jn} B_{ni}) = 0 \quad \text{for } j \neq i \quad \text{— (iv)}$$

Date
Page

So, if $B = [B_{ij}]$ then.

from.

(B), (D) & (E) we have $AB = I$, and hence

~~B is inverse~~

$A = B^{-1}$

~~* B is invertible~~ $\Rightarrow$ A is invertible

□.

Part 2

A is invertible $\Rightarrow$ Column span $\mathbb{R}^n$.

We prove, it's converse, i.e,

Column does not span $\mathbb{R}^n \Rightarrow$ A is Not invertible.

Suppose not, ie. A is invertible,

then $\exists\ B = [B_{ij}] \in M_n(\mathbb{R})$ s.t.

$AB = \mathbb{I}_n$.

$A = [A_{ij}]$.

on multiplying we find.

$(A_{i1} B_{1i} + \cdots + A_{in} B_{ni}) = 1 \quad \forall\ i = 1, \cdots n$

whereas

$(A_{j1} B_{1i} + \cdots + A_{jn} B_{ni}) = 0$

for $j \neq i$

Now,

column of $\mathbb{I}$ matrix are standard basis of $\mathbb{R}^n$, so, let they be $E_i\ \forall\ i \leq n$.

$\Rightarrow E_i = (A_{11} B_{1i} + \cdots + A_{1n} B_{ni}) E_1 + \cdots + (A_{n1} B_{1i} + \cdots A_{nn} B_{ni}) E_n$.

$= (A_{11} E_1 + \cdots + A_{n1} E_n) B_{1i} + \cdots + (A_{1n} E_1 + \cdots + A_{nn} E_n) B_{ni}$

$\therefore E_i = \sum_{k \geq n} C_i B_{ki}$, when

C_i denotes the column of A.

$C_i = \sum A_{ji} E_j$

So, as ~~B_{ii} = unique~~

The set of constants, ~~B_{ij}~~

B_{ki} = unique. for $i = 1, \dots, n$.

⇒) Any standard basis in $\mathbb{R}^n$ (E_i) can be expressed as linear combination of columns of A

⇒) columns span $\mathbb{R}^n$.

⟶⟵ (contradiction)

∴ Our assumption was wrong,

⇒ A is Not invertible, ∎

So, it's converse, is what we wished to prove

(2)

$$\begin{vmatrix} 2 & -2 & & & 0 \\ -1 & 5 & -2 & & \\ 0 & -2 & 5 & -2 & \\ & \ddots & \ddots & \ddots & \\ 0 & & & & \end{vmatrix} = |A|$$

$$\begin{vmatrix} -2 & 5 & -2 & 0 \\ 0 & -2 & 5 & -1 \\ 0 & 0 & -2 & 2 \end{vmatrix}_{n \times n}.$$

We do

(i) $R_i \to R_i/2$

(ii) $R_{i+1} \to R_i + R_{i+1}$ $\quad \forall\, i = \{1, \dots, n-2\}$.

$$\Rightarrow \frac{|A|}{2^{n-2}} = \begin{vmatrix} 1 & -1 & 0 & & \\ 0 & 2 & -1 & 0 & \\ 0 & 0 & 2 & -1 & \\ & & & & \end{vmatrix}$$

(i) $R_{n-1} \to R_{n-2} + R_{n-1}$.

$$\begin{vmatrix} 0 & 2 & -1 & 0 \\ 0 & -2 & 5 & -1 \\ 0 & 0 & -2 & 2 \end{vmatrix}$$

(ii) $\frac{R_n}{2}$.

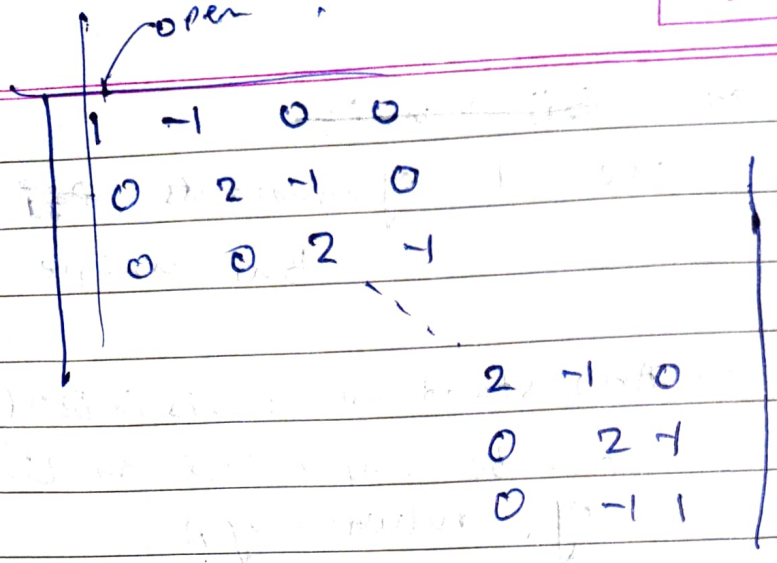
(iii) $R_{n-1} \to R_n + R_{n-1}$.

$$\frac{|A|}{2^{n-1}} = \begin{vmatrix} \ddots & 0 & 2 & -1 & 0 \\ & 0 & 0 & 3 & 0 \\ & 0 & 0 & -1 & 1 \end{vmatrix}$$

(i) $R_{n-1}/3$

(ii) $R_{n-1} \to R_{n-1} - R_n$.

Date
Page



$$\Rightarrow \frac{|A|}{3 \cdot 2^{n-1}} = \begin{vmatrix} 1 & -1 & 0 & 0 & \cdots \\ 0 & 2 & -1 & 0 & \cdots \\ 0 & 0 & 2 & -1 & \cdots \\ \vdots & & & \ddots & \\ & & & 2 & -1 & 0 \\ & & & 0 & 2 & -1 \\ & & & 0 & -1 & 1 \end{vmatrix}$$

$$= 2 \begin{vmatrix} 2 & -1 & 0 & \cdots \\ 0 & 2 & -1 & 0 \\ \vdots & & \ddots & \\ & & & 2 & -1 & 0 \\ & & & 0 & 2 & -1 \\ & & & 0 & -1 & 1 \end{vmatrix}$$

Make,

(i) $C_i / 2$

(ii) $C_{i+1} \to C_i + C_{i+1}$,

$\forall i \in \{1, \ldots n-3\}$.

$$= \frac{|A|}{3 \cdot 2^{n-1} \cdot 2^{n-3}}$$

$$= \begin{vmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ \vdots & & \ddots & \\ & & & 0 & 1 & 0 & 0 \\ & & & 0 & 0 & 2 & -1 \\ & & & 0 & 0 & -1 & 1 \end{vmatrix}$$

$$= 2 - (1) = 1$$

$$\Rightarrow \boxed{|A| = 3 \cdot 2^{2n-4}.}$$

Ans

MA2102 – Assignment -1

Name- SUSNATA CHATTOPADHYAY

Roll- 22MS077.

Group-

Date | 8/8/2023

Page |

③ Let v be the eigenvector, associated with eigenvalue 'λ' for A.

Then.

$$A = Av = \lambda v .$$

Let ~~$A = [a_{mn}]$~~ ~~$A = [a_{pq}]$~~.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots \\ a_{21} & a_{22} & a_{23} & \cdots \\ & & \vdots & \\ & 0 & a_{nn} & \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{bmatrix} = \lambda \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$k=2$:

$\therefore a_{21} v_1 + a_{22} v_2 + a_{23} v_3 + \cdots a_{2n} v_n = \lambda v_2$

$2 \to k$

$$\sum_{j=1}^{n} a_{kj} v_j = \lambda v_k . \quad \forall k \le n .$$

$$\Rightarrow \sum_{j=1, j \neq k}^{n} a_{kj} v_j = \lambda v_k - a_{kk} v_k$$

$$= v_k (\lambda - a_{kk}).$$

Now using Polygon inequality

$$|a_1 + a_2 + \cdots a_n| \le |a_1| + |a_2| + \cdots + |a_n| .$$

$$\Rightarrow \sum_{j \le n, j \neq k} |a_{kj}||v_j| \ge |v_k| |\lambda - a_{kk}| .$$

$$\Rightarrow \sum_{k=1}^{n} \sum_{kj=1, j \neq k}^{n} |a_{kj}||v_j| \ge \sum_{k=1}^{n} |v_k| |\lambda - a_{kk}| .$$

$$\Rightarrow \sum_{k \le n} \sum_{j \le n, j \neq k} |a_{jk}| |v_k| \ge \sum_{k \le n} |v_k| |\lambda - a_{kk}| .$$

Now, suppose that

$$|\lambda - a_{kk}| \not> \sum_{j\le n, j\neq k} |a_{jk}|.$$

$$\Rightarrow |v_k|\,|\lambda - a_{kk}| > \sum_{j\le n, j\neq k} |v_k|\,|a_{jk}|$$

$$\Rightarrow \sum_{k=1}^{n} |v_k|\,|\lambda - a_{kk}| > \sum_{k=1}^{n} \sum_{j=1, j\neq k}^{n} |a_{jk}|\,|v_k|$$

So, this contradicts our result!!

So, we can say

$\exists\ k \le n,\ k \in \mathbb{Z}^+$ s.t.

$$|\lambda - a_{kk}| \le \sum_{j=1, j\neq k}^{n} |a_{jk}|$$

(proved)

Ex-5 (i) $x \boxplus y$ = maximum of x and y.

$\forall\ x, y \in V.$, $V = \mathbb{R}$.

$\alpha \boxdot x = \alpha x$

$\forall\ \alpha \in \mathbb{R},\ x \in V.$

Is V a vector space over $\mathbb{R}$?

Ans. check 1 — Is $(V, \boxplus)$ a group?

(a) Associativity

choose $z, x, y \in V$.

Then, $x \boxplus (y \boxplus z) = x \boxplus (\text{Max}\{y, z\}) = \text{Max}\{x, y, z\}$

$= (\text{Max}\{x, y\}) \boxplus z$

$= (x \boxplus y) \boxplus z$.

✓

(b) Identity element

Suppose $\exists\ i_d \in V$, s.t.

$x \boxplus i_d = x$

$\Rightarrow \max(x, i_d) = x$

$\Rightarrow$ either $i_d < x$ OR $i_d = x$.

If $i_R = x$

Then choose y s.t. ~~$y \neq x$~~ $y < x$.

$\Rightarrow$ $i_R \boxplus y = \max(i_R, y) = \max(x, y) = x \neq y$

$\rightarrow \leftarrow$

If $i_R < x$ $\quad y \in \mathbb{R}$.

Then on $V = \mathbb{R}$, $\exists\ y$ s.t $y < i_R < x$.

$\Rightarrow$ $i_R \boxplus y = \max(i_R, y) = i_R \neq y$.

$\rightarrow \leftarrow$

So, No identity element exists over binary addition $\boxplus$ in V.

$\Rightarrow$ $(V, \boxplus)$ is not a group

$\Rightarrow$ V is not a vector space.

□.

(ii) V be $= \mathbb{R}^+$. Define binary operation, $\boxplus$.

$x \boxplus y = xy$

$\forall\ x, y \in V$.

$\alpha \boxdot x = x^\alpha$

$\forall \alpha \in \mathbb{R}$.

Show V is a vector space.

Ans Check 1 $(V, +)$ forms a Abelian group

(i) Associativity

Take $z, x, y \in V$

$z \boxplus (x \boxplus y) =$ ~~$xy = yx$~~ $=$

$= z \boxplus (xy) = (xy)z = (xz)y = (z \boxplus x) \boxplus y$

$= y \boxplus (z \boxplus x) = y \boxplus (x \boxplus z)$

✓.

(ii) Identity (i_R)

since $V = \mathbb{R}^+$ Take $x, 1 \in \mathbb{R} V$

$\therefore x \boxplus 1 = x = 1 \boxplus x$

$\therefore \exists\ i_R = 1$

(iii) Note $0 \notin V$

Then for each $x \in V$, $\exists\ z \in V$,

S.T

$x \boxplus z = xz = 1 = z \boxplus x$

We denote $z := \frac{1}{x}$ (since $x \neq 0$, $\frac{1}{x}$ exist)

∴ Additive Inverse Exists.

(iv) ~~Check~~ Take $a, b \in V$

$a \boxplus b = ab = ba = b \boxplus a$

So, binary addition is commutative

∴ $(V, +)$ forms an Abelian Group

Check-2 Now, we have to show Scalar Multiplication satisfies the following ~~axioms~~ properties.

(i) Associativity Take, $a, b \in \mathbb{F}, v, x \in V$

T.S $a \boxdot (b \boxdot x) = (ab) \boxdot x$

$a \boxdot (b \boxdot x) = a \boxdot (x^b)$ since $x^b \in V$ [V is closed under $\boxdot$]

$= (x^b)^a = x^{ab} = (ab) \boxdot x$

(ii) Distributivity.

T.S $(a+b) \boxdot v = (a \boxdot v) \boxplus (b \boxdot v)$

$(a+b) \boxdot v = v^{a+b} = v^a \cdot v^b = v^a \boxplus v^b$

$= (a \boxdot v) \boxplus (b \boxdot v)$

(iii) We have $1 \in F$. Then

$1 \boxdot v = v^1 = v \quad \forall \cdot v \in V$

∴ Multiplicative identity exists.

Hence, from Check 1 and Check 2, we conclude that

V is indeed a vector space. □