NUMBER SYSTEMS

1. WELL ORDERING PRINCIPLE (WOP)

Every non empty subset of $\mathbb{N}$ has a least element. i.e

Let, $S \subseteq \mathbb{N}$, $S \neq \phi$

Then $\exists\ m \in S$, such that $m \leq x$, $\forall x \in S$

2. PRINCIPLE OF INDUCTION (PI)

Let $S \subseteq \mathbb{N}$, which follows

(i) $1 \in S$

(ii) $k+1 \in S$, whenever $k \in S$

Then $\boxed{S = \mathbb{N}}$

3. PRINCIPLE OF STRONG INDUCTION (PSI)

Let $S \subseteq \mathbb{N}$, which follows

(i) $1 \in S$

(ii) $k+1 \in S$, whenever $\{1, 2, \dots k\} \subseteq S$ $\forall\ k \in \mathbb{N}$

Then, $\boxed{S = \mathbb{N}}$

Remark:



One is true iff other two are also true.

4. Every $n \in \mathbb{N}$, $n \geq 2$ is a product of primes (i.e has a prime divisor / factor).

5. INTEGERS

$$\mathbb{Z} = (\mathbb{N} \times \mathbb{N}) / \sim$$

where, $\sim$ is an equv. relation of $\mathbb{N} \times \mathbb{N}$ which follows, $\forall$ (m,n), $(p,q) \in \mathbb{N} \times \mathbb{N}$

$(m,n) \sim (p,q)$ if $m - n = p - q$

i.e, $[(m,n)] = [(p,q)]$

$$(\mathbb{N} \times \mathbb{N}) / \sim = \{[(1,j)] \mid j \geq 2\} \cup \{[(j,1)] \mid j \geq 2\} \cup \{[(1,1)]\}$$

$a, b \in \mathbb{Z}$

$a + b = [(m+p, n+q)]$

$a \cdot b = [(mp + nq, qm + np)]$

$(\mathbb{Z}, +, \cdot)$ → commutative Ring with identity

6. Cancellation

$\forall\ n, p, q \in \mathbb{N}$, $n + p = n + q \Rightarrow p = q$

Take $S(x) = n + x$, Prove by PI

Re 4/4/2023

7. RATIONAL NUMBER

$$\mathbb{Q} = \frac{\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})}{\sim}$$

where $\sim$ is an eq. rel^n on $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ which follows

$\forall\ m, p \in \mathbb{Z}$ & $n, q \in \mathbb{Z} - \{0\}$

$(m,n) \sim (p,q) \iff \frac{m}{n} = \frac{p}{q}$

$\Rightarrow mq = np$

$a, b \in \mathbb{Q}$

$a + b := [(mq + np, nq)]$

$a \cdot b := [(mp, nq)]$.

$(\mathbb{Q}, +, \cdot)$ is a field.

* PIGEON HOLE PRINCIPLE (PHP)

Let $n, r \in \mathbb{N}$ and $n > r$

If n objects are placed in 'r' boxes, then atleast one box will contain more than one object.

CALCULUS

1. LIMIT

Let, $-\infty < a < b < \infty$; $f : (a,b) \to \mathbb{R}$

Let $x_0 \in [a,b]$ and $L \in \mathbb{R}$

Then

$$L := \lim_{x \to x_0} f(x)$$

exists, if for every $\epsilon > 0$, $\exists\ \delta > 0$ such that, $\forall\ x \in (a,b)$ satisfying

$$0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \epsilon$$

2. CONTINUITY

(.....)

We say f is continuous at x_0, if for every $\epsilon > 0$, $\exists\ \delta > 0$ such that, $\forall\ x \in (a,b)$ whenever

$$|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon$$

LEMMA

f is continuous at x_0, if and only if

(i) $\lim_{x \to x_0} f(x)$ exists, &

(ii) $\lim_{x \to x_0} f(x) = f(x_0)$

3. DIFFERENTIABILITY

(....) $x_0 \in [a,b]$. $f : [a,b] \to \mathbb{R}$

we say f is differentiable at x_0, if the limit exists and

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

7. **Successor function** (Assumption)

The map $S: \mathbb{N} \to \mathbb{N}$ defined as

$S(n) = n+1 \quad \forall\, n \in \mathbb{N}$ has

properties

(i) S is $1-1$

(ii) $S(n) \neq 1, \quad \forall\, n \in \mathbb{N}$.

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* 4. **Chain Rule**.

Let, $-\infty < a < b < \infty, \quad -\infty < c < d < \infty$

$f: [a,b] \to [c,d]$

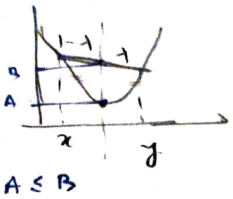
$g: [c,d] \to \mathbb{R}$.

If f is differentiable at x_0, $x_0 \in [a,b]$

and g is differentiable at $f(x_0)$

Then $gof: [a,b] \to \mathbb{R}$ is differentiable at x_0 and $(gof)'(x_0) = g'(f(x_0))\, f'(x_0)$.

CONVEX FUNCTIONS



A ≤ B

1. Defn.

Let $I \subseteq \mathbb{R}$, A function $f: I \to \mathbb{R}$ is said to be convex if, $\forall\ x, y \in I$ and $\lambda \in [0,1]$

$$f(\lambda x + (1-\lambda) y) \leq \lambda f(x) + (1-\lambda) f(y)$$

We call f concave, if $-f$ is convex.

2. RISING SLOPE THEOREM.

Let $I \subseteq \mathbb{R}$, $f: I \to \mathbb{R}$. Then f is a convex function iff $\forall\ x, y, z \in I$,

$x > y > z \to$

$$\frac{f(x) - f(y)}{x - y} \geq \frac{f(y) - f(z)}{y - z}$$

Take $y = \lambda x + (1-\lambda) z$, $\lambda \in (0,1)$

3. THEOREM.

Let $I \subseteq \mathbb{R}$. $f: I \to \mathbb{R}$., f is differentiable. Then function is convex if and only if, f' is increasing, ie

$$f'(x) \geq f'(y) \quad \forall\ x, y \in I,\ x \geq y.$$

COROLLARY

Let $f: I \to \mathbb{R}$ be twice differentiable, then f is convex $\iff f''(x) \geq 0$ $\forall\ x \in I$.

INEQUALITY

1. JENSEN'S THEOREM

Let $I \subseteq \mathbb{R}$, be an interval, $f: I \to \mathbb{R}$. Then f is convex if and only if, $\forall\ n \in \mathbb{N}$ $n \geq 2$, and for all $\lambda_1, \dots \lambda_n \in [0,1]$ & $x_1, x_2 \dots x_n \in I$ satisfying

$$\lambda_1 + \lambda_2 + \dots + \lambda_n = 1,$$

$$f(\lambda_1 x_1 + \lambda_2 x_2 + \dots \lambda_n x_n) \leq \lambda_1 f(x_1) + \dots + \lambda_n f(x_n)$$

i.e $$f\left(\sum_{i=1}^{n} \lambda_i x_i\right) \leq \sum_{i=1}^{n} \lambda_i f(x_i)$$

2. P^{th} Power Inequality.

Let $1 \leq p < \infty$ and let $a_1, \dots a_n > 0$

Then $$\left(\frac{a_1^p + \dots + a_n^p}{n}\right) \geq \left(\frac{a_1 + \dots + a_n}{n}\right)^p$$

Proof

PT on $f: (0,\infty) \to \mathbb{R}$, $f(x) = x^p$

3. AM-GM-HM Inequality

Let $a_1, \dots, a_n > 0$ $n \in \mathbb{N}, \geq 2$

Then

$$\frac{a_1 + a_2 + \dots a_n}{n} \geq (a_1 a_2 \dots a_n)^{1/n} \geq \frac{n}{\frac{1}{a_1} + \dots + \frac{1}{a_n}}$$

i.e $$\frac{\sum_1^n a_i}{n} \geq \left(\prod_1^n a_i\right)^{1/n} \geq \frac{n}{\sum_1^n \frac{1}{a_i}}$$

4. YOUNG'S INEQUALITY.

Let $a, b > 0$ and let $p, q > 1$ be such that

$$\frac{1}{p} + \frac{1}{q} = 1$$

Then,

$$\frac{1}{p} a^p + \frac{1}{q} b^q \geq a b$$

5. HÖLDER'S INEQUALITY.

Let $a_1, \dots a_n, b_1, \dots, b_n \geq 0$ and let $1 < p, q < \infty$ satisfy

$$\frac{1}{p} + \frac{1}{q} = 1$$

Then $(a_1 b_1 + \dots a_n b_n) \leq (a_1^p + \dots a_n^p)^{1/p} \cdot (b_1^q + \dots + b_n^q)^{1/q}$

i.e $$\left(\sum_1^n a_i b_i\right) \leq \left(\sum_1^n a_i^p\right)^{1/p} \left(\sum_1^n b_i^q\right)^{1/q}$$

If, $p = q = 2$, we get Cauchy Schwarz Inequality.

$$(a_1 b_1 + \dots + a_n b_n) \leq (a_1^2 + a_2^2 + \dots a_n^2)^{1/2} (b_1^2 + \dots + b_n^2)^{1/2}$$

6. MINKOWSKI INEQUALITY

Let $a_1, \dots a_n, b_1, \dots b_n \geq 0$ & let $p \in [1,\infty)$

Then, $$\left[\sum_{i=1}^n (a_i + b_i)^p\right]^{1/p} \leq \left[\sum_1^n a_i^p\right]^{1/p} + \left[\sum_1^n b_i^p\right]^{1/p}$$

It gives us Δ ineq for different dimensions. $p = 2 \to$ Euclidean.

4. C&D

Let $-\infty < a < b < \infty$ $f:[a,b] \to \mathbb{R}$ and $x_0 \in [a,b]$

f is differentiable at x_0, implies f is also continuous at x_0

BUT, converse is NOT True.

Proof by

$$\phi(x) = \begin{cases} \dfrac{f(x)-f(x_0)}{x-x_0} & x \neq x_0 \\ f'(x_0) & x = x_0 \end{cases}$$

MEAN VALUE THEOREMS (MVT)

1. ROLLES THEOREM :

Let, $-\infty < a < b < \infty$ $f:[a,b] \to \mathbb{R}$ satisfy

(i) $f(x)$ is cont. on $[a,b]$

(ii) $f(x)$ is diff on (a,b)

(iii) $f(a) = f(b)$

then $\exists$ atleast one $c \in (a,b)$ such that $\boxed{f'(c) = 0}$

2. LAGRANGES THEOREM

(- - - -)

i) $f(x)$ is cont. on $[a,b]$

ii) $f(x)$ is diff. on (a,b)

then $\exists$ $c \in (a,b)$ such that

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

$$\Rightarrow \quad f(b) = f(a) + f'(c)(b-a)$$

CAUCHY THEOREM (Generalise)

Let $f, g : [a,b] \to \mathbb{R}$ be

(i) cont. on $[a,b]$

(ii) diff on (a,b)

then for some $c \in (a,b)$

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

$$\Rightarrow \quad g'(c)\{f(b)-f(a)\} = f'(c)\{g(b)-g(a)\}$$

4. L'HÔPITAL RULE

Let $a < c < b$ & $f, g : (a,b) \to \mathbb{R}$ be differentiable in $(a,b) \setminus \{c\}$

Let us suppose.

(i) $f(c) = g(c) = 0$

(ii) $g(x) \neq 0$, $\forall\, x \in (a,b)$, $x \neq c$

(iii) $g'(x) \neq 0$, $\forall\, x \in (a,b)$, $x \neq c$

If, $\lim_{x \to c} \frac{f'(x)}{g'(x)}$ exists. Then.

$\lim_{x \to c} \frac{f(x)}{g(x)}$ also exist.

And

$$\boxed{\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}}$$

5. 2nd ORDER MVT (Taylor's).

Let $-\infty < a < c < b < \infty$ and $f:[a,b] \to \mathbb{R}$ satisfy

(i) f, f' is continuous in $[a,b]$

(ii) f' is differentiable in $(a,b]$

Then $\exists$ $c \in (a,b)$ such that

$$f(b) = f(a) + f'(a)(b-a) + \frac{1}{2} f''(c)(b-a)^2$$

Apply Rolle on

$$\varphi(x) = f(b) - f(x) - f'(x)(b-x) - A(b-x)^2.$$

A: such that $\varphi(a) = \varphi(b)$

MAXIMA AND MINIMA

1. Defn

Let $I \subseteq \mathbb{R}$ be an interval, $f: I \to \mathbb{R}$, $c \in I$ we say that c is a part of local max^m/min^m if for some $\delta > 0$

(A) local strict max^m

$f(c) > f(x)$ $\forall\, x \in (c-\delta, c+\delta) \cap I$

(B) local strict min^m

$f(c) < f(x)$ $\forall\, x \in (c-\delta, c+\delta) \cap I$.

Let $a < b$, $f:(a,b) \to \mathbb{R}$. Let $c \in (a,b)$ be a point of local min/max & f is differ^n at c, then $\boxed{f'(c) = 0}$.

Now if

(i) f, f', f'' are continuous in (a,b)

(ii) $f'(c) = 0$ & $f''(c) \neq 0$ for some $c \in (a,b)$

Then c is a point of local

- Min^m if $f''(c) > 0$
- Max^m if $f''(c) < 0$

SEM-2

SEQUENCE AND SERIES

AC

1. SEQUENCE

A sequence $\{x_n\}$ in $\mathbb{R}$ is a map from $\mathbb{N} \to \mathbb{R}$

$x_n = f(n)$

$1 \mapsto x_1$
$2 \mapsto x_2$
$\vdots$
$n \mapsto x_n$

The range of a real sequence $\{f(n)\}$ is a subset of $\mathbb{R}$

Note $x_n = \frac{1}{n-5}$, $n \in \mathbb{N}$

It is not sequence, as $f : \mathbb{N} \to \mathbb{R}$ Not followed.

2. Archimedian Property.

For every $\varepsilon \in \mathbb{R}$, $\exists\ K \in \mathbb{N}$ s.t $\varepsilon < K$

3. Boundary.

3.1 Least Upper Bound (Lub)

The lub of a seqⁿ $\{x_n\}$ is a real number M, such that

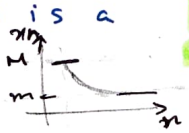
Think!

(i) $x_n \leq M \quad \forall\ n \in \mathbb{N}$

(ii) $\forall$ (small) $\varepsilon > 0$, $\exists\ K \in \mathbb{N}$ s.t $x_K > M - \varepsilon$

3.2 Greatest Lower Bound (glb)

The glb of a seqⁿ $\{x_n\}$ is a real number (m), s.t



(i) $x_n \geq m \quad \forall\ n \in \mathbb{N}$

(ii) $\forall$ (small) $\varepsilon > 0$, $\exists\ K \in \mathbb{N}$ s.t $x_K < m + \varepsilon$

∴ A real sequence, $\{f(n)\}$ is said to be bounded sequence if $\exists$ a real number G, g such that

$g \leq f(n) \leq G \quad \forall\ n \in \mathbb{N}$

4. LIMIT of a seqⁿ.

$\lim \{x_n\} = \ell$ means the sequence converges to ℓ $(\in \mathbb{R})$ if given any $\varepsilon > 0$, $\exists\ K \in \mathbb{N}$ such that,

$|x_n - \ell| < \varepsilon$

$\forall\ n \geq K$

5. THEOREMS

1) Converging seqⁿ has unique limit

2) If seqⁿ has no limit ⇒ Seqⁿ is divergent.

3) Not convergent ⇏ Divergence.

4) A convergent seqⁿ has both lower & upper bound.

5) But a bounded seqⁿ may NOT be convergent (Eg- $\sin(\frac{n\pi}{2})$ $n \in \mathbb{N}$)

$A \to B$ is true, then $\sim B \to \sim A$ Contra(+) also true.

6) Let $x_n \to x$ & $y_n \to y$ and $c \in \mathbb{R}$

- $x_n \pm y_n \to x \pm y$
- $c\{x_n\} \to cx$
- $x_n y_n \to xy$
- If $\{y_n\} \neq 0, \forall\ n \geq 1$ & $y \neq 0$ Then $\{\frac{1}{y_n}\} \to \frac{1}{y}$
- If $x_n \to x \Rightarrow |x_n| \to |x|$ converse NOT true.
- $x_n \to 0 \iff |x_n| \to 0$
- $x_n \to x \iff x_n - x \to 0$ $\iff |x_n - x| \to 0$.

Note • $x_n \geq 0$; $x_n \to x$ then $\sqrt{x_n} \to \sqrt{x}$ (Real)

If $\{x_n\}, \{y_n\}$ are convergent seqⁿ it follows →

- if $x_n \geq 0\ \forall\ n \geq m (\in \mathbb{N})$ Then $\lim x_n \geq 0$
- if $x_n \geq y_n\ \forall\ n \geq m\ (\in \mathbb{N})$ $\lim x_n \geq \lim y_n$.
- if $a \leq x_n \leq b$, $\forall\ n \geq m \in \mathbb{N}$ $\Rightarrow \lim\{x_n\} \in [a, b]$.

* Let $x_n \leq y_n \leq z_n \quad \forall\ n \in \mathbb{N}$ & s.t, $\lim x_n = \lim z_n$ (All CONV)
⇒ y_n is also conv & $\lim x_n = \lim y_n = \lim z_n$

** 6) DIVERGING CONDS

backcal: from here

- $\{x_n\}$ diverges to $+\infty$ iff $\forall\ M > 0\ \exists\ N \in \mathbb{N}$, s.t $\forall\ n \geq N$, $x_n > M$
- $\{x_n\}$ diverges to $-\infty$ iff $\forall\ M < 0\ \exists\ K \in \mathbb{N}$ s.t $x_n < M\ \forall\ n \geq K$



7) TESTS FOR DIVERGENCE

7.1) RATIO TEST

Let x_n be a positive real no.

s.t $\lim \left\{ \frac{x_{n+1}}{x_n} \right\} = x$

(i) if $0 \leq x < 1$ then $\lim x_n = 0$

(ii) if $x > 1$ then $\lim x_n = \infty$ (Diverges)

SERIES

If $X := (x_n)$ is a sequence in $\mathbb{R}$, then series generated by X is the sequence $S := (S_n)$ defined by

$S_1 = x_1$

$S_2 = x_1 + x_2$

$\vdots$

$S_n = \sum_{i=1}^{n} x_i$

$\lim S_n$ → osci / conv / $\infty/-\infty$

7.2) INEQUALITY TEST:

Let $\sum x_n$ & $\sum y_n$ be two series of $\mathbb{R}^+$,

& $\exists$ an $m \in \mathbb{N}$ s.t $\boxed{x_n \leq k y_n}$

$\forall n \geq m$. k being a fixed positive Real.

Then

① $\sum y_n$ converges $\Rightarrow$ $\sum x_n$ converges.

② $\sum x_n$ diverges $\Rightarrow$ $\sum y_n$ diverges.

7.3) LIMIT TEST:

If $\lim \frac{x_n}{y_n} = l$ where l is a non-zero Finite No. Then,

- $\sum x_n$ converges $\Leftrightarrow$ $\sum y_n$ converges
- $\sum x_n$ diverges $\Leftrightarrow$ $\sum y_n$ diverges.

8) USEFUL RESULTS:

8.1 For Geometric series, $\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + \cdots$

$$S_n = \frac{1}{1-r} - \frac{r^{n+1}}{1-r}$$

Case 1 $|r| < 1$

$\lim r^{n+1} = 0$ $\therefore \lim S_n = \frac{1}{1-r}$ (conv).

Case 2 $r = 1$

$\lim S_n = \infty$ (Diverge) $S_n = n$

Case 3 $r = -1$

$\lim S_n \to 0$ / $\lim S_n \to 1$ Does NOT converge

Case 4 $r < -1$

$\lim S_n \to$ Diverge.

8.2 $S_n = \sum_{n=1}^{\infty} \frac{1}{n^p}$

- $p \leq 0$ $\sum_{n=1}^{\infty} n^{-p} \to$ Diverges.
- $p = 1$ $\lim S = \sum \frac{1}{n} \to$ Diverges
- $0 < p < 1$ $\lim S \to$ Diverges
- $p > 1$ $\lim S \to$ converges.

8.4 $n! > n^2$ $\forall n \geq 4$ | $\sin x < x < \tan x$, $\log x < x < e^x$, $2^n < (n-1)!$ $\forall n \geq 6$